Monatsh. Math. 171 (2013) (3-4) 341–350

# A construction of integer-valued polynomials with prescribed sets of lengths of factorizations

Sophie Frisch[1]

**Abstract.** For an arbitrary finite set $S$ of natural numbers greater 1, we construct $f \in \mathrm{Int}(\mathbb{Z}) = \{g \in \mathbb{Q}[x] \mid g(\mathbb{Z}) \subseteq \mathbb{Z}\}$ such that $S$ is the set of lengths of $f$, i.e., the set of all $n$ such that $f$ has a factorization as a product of $n$ irreducibles in $\mathrm{Int}(\mathbb{Z})$. More generally, we can realize any finite multi-set of natural numbers greater 1 as the multi-set of lengths of the essentially different factorizations of $f$.

MSC 2000: primary 13A05, secondary 13B25, 13F20, 20M13, 11C08.

## 1. *Introduction*

Non-unique factorization has long been studied in rings of integers of number fields, see the monograph of Geroldinger and Halter-Koch [5]. More recently, non-unique factorization in rings of polynomials has attracted attention, for instance in $\mathbb{Z}_{p^n}[x]$, cf. [4], and in the ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z}) = \{g \in \mathbb{Q}[x] \mid g(\mathbb{Z}) \subseteq \mathbb{Z}\}$ (and its generalizations) [1, 3].

We show that every finite set of natural numbers greater 1 occurs as the set of lengths of factorizations of an element of $\mathrm{Int}(\mathbb{Z})$ (Theorem 9 in section 4).

Our proof is constructive, and allows multiplicities of lengths of factorizations to be specified. For example, given the multiset $\{2, 2, 2, 5, 5\}$, we construct a polynomial that has three different factorizations into 2 irreducibles and two different factorizations into 5 irreducibles, and no other factorizations. Perhaps a quick review of the vocabulary of factorizations is in order:

**Notation and Conventions.** *$R$ denotes a commutative ring with identity. An element $r \in R$ is called* irreducible *in $R$ if $r$ is a non-zero non-unit such that $r = ab$ with $a, b \in R$ implies that $a$ or $b$ is a unit. A* factorization *of $r$ in $R$ is an expression $r = s_1 \dots s_n$ of $r$ as a product of irreducible elements in $R$. The number $n$ of irreducible factors is called the* length *of the factorization. The* set of lengths *$\mathcal{L}(r)$ of $r \in R$ is the set of all natural numbers $n$ such that $r$ has a factorization of length $n$ in $R$.*

*$R$ is called* atomic *if every non-zero non-unit of $R$ has a factorization in $R$. If $R$ is atomic, then for every non-zero non-unit $r \in R$ the* elasticity of $r$ *is defined*

[1] Supported by FWF (Austrian Science Fund) grant P23245

*as*

$$\rho(r) = \sup\{\frac{m}{n} \mid m, n \in \mathcal{L}(r)\}$$

*and the elasticity of $R$ is $\rho(R) = \sup_{r\in R'}(\rho(r))$, where $R'$ is the set of non-zero non-units of $R$. An atomic domain $R$ is called fully elastic if every rational number greater than 1 occurs as $\rho(r)$ for some non-zero non-unit $r \in R$.*

*Two elements $r, s \in R$ are called associated in $R$ if there exists a unit $u \in R$ such that $r = us$. Two factorizations of the same element $r = r_1 \cdot \ldots \cdot r_m = s_1 \cdot \ldots \cdot s_n$ are called essentially the same if $m = n$ and, after re-indexing the $s_i$, $r_j$ is associated to $s_j$ for $1 \le j \le m$. Otherwise, the factorizations are called essentially different.*

## 2. *Review of factorization of integer-valued polynomials*

In this section we recall some elementary properties of $\mathrm{Int}(\mathbb{Z})$ and the fixed divisor $\mathrm{d}(f)$, to be found in [1], [2] and [3]. The reader familiar with integer-valued polynomials is encouraged to skip to section 3.

**Definition.** *For $f \in \mathbb{Z}[x]$,*

(i) *the content $\mathrm{c}(f)$ is the ideal of $\mathbb{Z}$ generated by the coefficients of $f$,*

(ii) *the fixed divisor $\mathrm{d}(f)$ is the ideal of $\mathbb{Z}$ generated by the image $f(\mathbb{Z})$.*

*By abuse of notation we will identify the principal ideals $\mathrm{c}(f)$ and $\mathrm{d}(f)$ with their non-negative generators. Thus, for $f = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}[x]$,*

$$\mathrm{c}(f) = \gcd(a_k \mid k = 0, \ldots, n) \quad \text{and} \quad \mathrm{d}(f) = \gcd(f(c) \mid c \in \mathbb{Z}).$$

*A polynomial $f \in \mathbb{Z}[x]$ is called primitive if $\mathrm{c}(f) = 1$.*

Recall that a primitive polynomial $f \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ if and only if it is irreducible in $\mathbb{Q}[x]$. Similarly, $f \in \mathbb{Z}[x]$ with $\mathrm{d}(f) = 1$ is irreducible in $\mathbb{Z}[x]$ if and only if it is irreducible in $\mathrm{Int}(\mathbb{Z})$.

We denote $p$-adic valuation by $v_p$. Almost everything that we need to know about the fixed divisor follows immediately from the fact that

$$v_p(\mathrm{d}(f)) = \min_{c\in\mathbb{Z}}(v_p(f(c))).$$

In particular, it is easy to deduce that for any $f, g \in \mathbb{Z}[x]$,

$$\mathrm{d}(f)\mathrm{d}(g) \bigm| \mathrm{d}(fg).$$

Unlike $\mathrm{c}(f)$, which satisfies $\mathrm{c}(f)\mathrm{c}(g) = \mathrm{c}(fg)$, $\mathrm{d}(f)$ is not multiplicative: $\mathrm{d}(f)\mathrm{d}(g)$ is in general a proper divisor of $\mathrm{d}(fg)$.

**Remark 1.**

(i) *Every non-zero polynomial $f \in \mathbb{Q}[x]$ can be written in a unique way as*

$$f(x) = \frac{ag(x)}{b} \quad \text{with} \quad g \in \mathbb{Z}[x],\ c(g) = 1, \quad a, b \in \mathbb{N},\ \gcd(a, b) = 1.$$

(ii) *When expressed as in* (i), *$f$ is in* $\mathrm{Int}(\mathbb{Z})$ *if and only if $b$ divides* $\mathrm{d}(g)$.

(iii) *For non-constant $f \in \mathrm{Int}(\mathbb{Z})$ expressed as in* (i) *to be irreducible in* $\mathrm{Int}(\mathbb{Z})$ *it is necessary that $a = 1$ and $b = \mathrm{d}(g)$.*

*Proof.* (i) and (ii) are easy. Ad (iii). Note that the only units in $\mathrm{Int}(\mathbb{Z})$ are $\pm 1$. By (ii), $b$ divides $\mathrm{d}(g)$. Let $\mathrm{d}(g) = bc$. Then $f$ factors as $a \cdot c \cdot (g/bc)$, where $(g/bc)$ is non-constant and $ac$ is a unit only if $a = c = 1$. $\square$

**Remark 2.**

(i) *Every non-zero polynomial $f \in \mathbb{Q}[x]$ can be written in a unique way (up to the sign of $a$ and the signs and indexing of the $g_i$) as*

$$f(x) = \frac{a}{b} \prod_{i \in I} g_i(x),$$

*with $g_i$ primitive and irreducible in $\mathbb{Z}[x]$ for $i \in I$ (a finite set) and $a \in \mathbb{Z}$, $b \in \mathbb{N}$ with $\gcd(a, b) = 1$.*

(ii) *A non-constant polynomial $f \in \mathrm{Int}(\mathbb{Z})$ expressed as in* (i) *is irreducible in* $\mathrm{Int}(\mathbb{Z})$ *if and only if $a = \pm 1$, $b = \mathrm{d}(\prod_{i \in I} g_i)$, and there do not exist $\emptyset \neq J \subsetneq I$ and $b_1, b_2 \in \mathbb{N}$ with $b_1 b_2 = b$ and $b_1 = \mathrm{d}(\prod_{i \in J} g_i)$, $b_2 = \mathrm{d}(\prod_{i \in I \setminus J} g_i)$.*

(iii) $\mathrm{Int}(\mathbb{Z})$ *is atomic.*

(iv) *Every non-zero non-unit $f \in \mathrm{Int}(\mathbb{Z})$ has only finitely many factorizations into irreducibles in* $\mathrm{Int}(\mathbb{Z})$.

*Proof.* Ad (ii). If $f$ is irreducible, the conditions on $f$ follow from Remark 1 (ii) and (iii). Conversely, if the conditions hold, what chance does $f$ have to be reducible? By Remark 1 (ii), we cannot factor out a non-unit constant, because no proper multiple of $b$ divides $\mathrm{d}(\prod_{i \in I} g_i)$. Any non-constant irreducible factor would, by Remark 1 (iii), be of the kind $(\prod_{i \in J} g_i)/b_1$ with $b_1 = \mathrm{d}(\prod_{i \in J} g_i)$, and its co-factor would be $(\prod_{i \in I \setminus J} g_i)/b_2$ with $b_1 b_2 = b$ and $b_2$ a divisor of $\mathrm{d}(\prod_{i \in I \setminus J} g_i)$. Also, $b_2$ could not be a proper divisor of $\mathrm{d}(\prod_{i \in I \setminus J} g_i)$, because otherwise $b_1 b_2 = b$ would be a proper divisor of $\prod_{i \in I} g_i$. So, the existence of a non-constant irreducible factor would imply the existence of $J$ and $b_1, b_2$ of the kind we have excluded.

Ad (iii). With $f(x) = ag(x)/b$, $g = \prod_{i \in I} g_i$ as in (i), $\mathrm{d}(g) = cb$ for some $c \in \mathbb{N}$, and $f(x) = acg(x)/\mathrm{d}(g)$ with $g(x)/\mathrm{d}(g) \in \mathrm{Int}(\mathbb{Z})$. We can factor $ac$ into irreducibles in $\mathbb{Z}$, which are also irreducible in $\mathrm{Int}(\mathbb{Z})$. Either $g(x)/\mathrm{d}(g)$ is irreducible, or (ii) gives an expression as a product of two non-constant factors of smaller degree. By iteration we arrive at a factorization of $g(x)/\mathrm{d}(g)$ into irreducibles.

Ad (iv). Let $f \in \mathrm{Int}(\mathbb{Z}) = (ag(x)/b)$ with $g = \prod_{i \in I} g_i$ as in (i). Then all factorizations of $f$ are of the form, for some $c \in \mathbb{N}$ such that $bc$ divides $\mathrm{d}(g)$,

$$f = a_1 \dots a_n c_1 \dots c_m \prod_{j=1}^{k} \frac{\prod_{i \in I_j} g_i}{d_j},$$

where $a = a_1 \dots a_n$ and $c = c_1 \dots c_m$ are factorizations into primes in $\mathbb{Z}$, $I = I_1 \cup \dots \cup I_k$ is a partition of $I$ into non-empty sets, $d_1 \dots d_k = bc$, $d_j = \mathrm{d}(\prod_{i \in I_j} g_i)$. There are only finitely many such expressions. $\square$

**Remark 3.**

(i) *The binomial polynomials*

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} \quad \text{for} \quad n \geq 0$$

*are a basis of* $\mathrm{Int}(\mathbb{Z})$ *as a free* $\mathbb{Z}$*-module.*

(ii) $n!f \in \mathbb{Z}[x]$ *for every* $f \in \mathrm{Int}(\mathbb{Z})$ *of degree at most* $n$.

(iii) *Let* $f \in \mathbb{Z}[x]$ *primitive,* $\deg f = n$ *and* $p$ *prime. Then*

$$v_p(\mathrm{d}(f)) \leq \sum_{k \geq 1} \left[\frac{n}{p^k}\right] = v_p(n!).$$

*In particular, if* $p$ *divides* $\mathrm{d}(f)$ *then* $p \leq \deg f$.

*Proof.* Ad (i). The binomial polynomials are in $\mathrm{Int}(\mathbb{Z})$ and they form a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$. If a polynomial in $\mathrm{Int}(\mathbb{Z})$ is written as a $\mathbb{Q}$-linear combination of binomial polynomials then an easy induction shows that the coefficients must be integers. (ii) follows from (i).

Ad (iii). Let $g = f/d(f)$. Then $g \in \mathrm{Int}(\mathbb{Z})$ and $\mathrm{d}(f)\mathbb{Z} = (\mathbb{Z}[x] :_{\mathbb{Z}} g)$. Since $n! \in (\mathbb{Z}[x] :_{\mathbb{Z}} g)$ by (ii), $\mathrm{d}(f)$ divides $n!$. $\square$

## 3. *Useful Lemmata*

**Lemma 4.** *Let $p$ be a prime, $I \neq \emptyset$ a finite set and for $i \in I$, $f_i \in \mathbb{Z}[x]$ primitive and irreducible in $\mathbb{Z}[x]$ such that $\mathrm{d}(\prod_{i\in I} f_i) = p$. Let*

$$g(x) = \frac{\prod_{i\in I} f_i}{p}.$$

*Then every factorization of $g$ in $\mathrm{Int}(\mathbb{Z})$ is essentially the same as one of the following:*

$$g(x) = \frac{\prod_{j\in J} f_j}{p} \cdot \prod_{i\in I\setminus J} f_i,$$

*where $J \subseteq I$ is minimal such that $\mathrm{d}(\prod_{i\in J} f_j) = p$.*

*Proof.* Follows from Remark 1 (iii) and the fact that $\mathrm{d}(f)\mathrm{d}(h)$ divides $\mathrm{d}(fh)$ for all $f, h \in \mathbb{Z}[x]$. □

The following two easy lemmata are constructive, since the Euclidean algorithm makes the Chinese Remainder Theorem in $\mathbb{Z}$ effective.

**Lemma 5.** *For every prime $p \in \mathbb{Z}$, we can construct a complete system of residues mod $p$ that does not contain a complete system of residues modulo any other prime.*

*Proof.* By the Chinese Remainder Theorem we solve, for each $k = 1, \ldots, p$ the system of congruences $s_k = k \bmod p$ and $s_k = 1 \bmod q$ for every prime $q < p$. □

**Lemma 6.** *Given finitely many non-constant monic polynomials $f_i \in \mathbb{Z}[x]$, $i \in I$, we can construct monic irreducible polynomials $F_i \in \mathbb{Z}[x]$, pairwise non-associated in $\mathbb{Q}[x]$, with $\deg F_i = \deg f_i$, and with the following property:*

*Whenever we replace some of the $f_i$ by the corresponding $F_i$, setting $g_i = F_i$ for $i \in J$ ($J$ an arbitrary subset of $I$) and $g_i = f_i$ for $i \in I \setminus J$, then for all $K \subseteq I$,*

$$\mathrm{d}(\prod_{i\in K} g_i) = \mathrm{d}(\prod_{i\in K} f_i).$$

*Proof.* Let $n = \sum_{i\in I} \deg f_i$. Let $p_1, \ldots, p_s$ be all the primes with $p_i \leq n$, and set $\alpha_i = v_{p_i}(n!)$. Let $q > n$ be a prime. For each $i \in I$, we find by the Chinese Remainder Theorem the coefficients of a polynomial $\varphi_i \in (\prod_{k=1}^{s} p_k^{\alpha_k})\mathbb{Z}[x]$ of smaller degree than $f_i$, such that $F_i = f_i + \varphi_i$ satisfies Eisenstein's irreducibility criterion with respect to the prime $q$. Then, with respect to some linear ordering of $I$, if $F_i$ happens to be associated in $\mathbb{Q}[x]$ to any $F_j$ of smaller index, we add a suitable non-zero integer divisible by $q^2 \prod_{k=1}^{s} p_k^{\alpha_k}$ to $F_i$, to make $F_i$ non-associated in $\mathbb{Q}[x]$ to all $F_j$ of smaller index.

The statement about the fixed divisor follows, because for every $c \in \mathbb{Z}$ and every prime $p_i$ that could conceivably divide the fixed divisor,

$$\prod_{i \in K} (g_i(c)) \equiv \prod_{i \in K} (f_i(c)) \mod p_i^{\alpha_i},$$

where $p_i^{\alpha_i}$ is the highest power of $p_i$ that can divide the fixed divisor of any monic polynomial of degree at most $n$. $\square$

## 4. *Constructing polynomials with prescribed sets of lengths*

We precede the general construction by two illustrative examples of special cases, corresponding to previous results by Cahen, Chabert, Chapman and McClain.

**Example 7.** *For every $n \geq 0$, we can construct $H \in \mathrm{Int}(\mathbb{Z})$ such that $H$ has exactly two essentially different factorizations in $\mathrm{Int}(\mathbb{Z})$, one of length 2 and one of length $n+2$.*

*Proof.* Let $p > n+1$, $p$ prime. By Lemma 5 we construct a complete set $a_1, \ldots, a_p$ of residues mod $p$ in $\mathbb{Z}$ that does not contain a complete set of residues mod any prime $q < p$. Let

$$f(x) = (x-a_2)(x-a_3)\ldots(x-a_p) \quad \text{and} \quad g(x) = (x-a_{n+2})(x-a_{n+3})\ldots(x-a_p).$$

By Lemma 6, we construct monic irreducible polynomials $F, G \in \mathbb{Z}[x]$, not associated in $\mathbb{Q}[x]$, with $\deg F = \deg f$, $\deg G = \deg g$, such that any product of a selection of polynomials from $(x-a_1), \ldots, (x-a_{n+1}), f(x), g(x)$ has the same fixed divisor as the corresponding product with $f$ replaced by $F$ and $g$ by $G$.

Let

$$H(x) = \frac{F(x)(x-a_1)\ldots(x-a_{n+1})G(x)}{p}.$$

By Lemma 4, $H$ factors into two irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$

$$H(x) = F(x) \cdot \frac{(x-a_1)\ldots(x-a_{n+1})G(x)}{p}$$

or into $n+2$ irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$

$$H(x) = \frac{F(x)(x-a_1)}{p} \cdot (x-a_2)(x-a_3)\ldots(x-a_{n+1})G(x).$$

$\square$

**Corollary.** (Cahen and Chabert [1]). $\rho\,(\mathrm{Int}(\mathbb{Z})) = \infty$.

**Example 8.** *For $1 \le m \le n$, we can construct a polynomial $H \in \mathrm{Int}(\mathbb{Z})$ that has in $\mathrm{Int}(\mathbb{Z})$ a factorization into $m+1$ irreducibles and an essentially different factorization into $n+1$ irreducibles, and no other essentially different factorization.*

*Proof.* Let $p > mn$ be prime, $s = p - mn$. By Lemma 5 we construct a complete system of residues $R$ mod $p$ that does not contain a complete system of residues for any prime $q < p$. We index $R$ as follows:

$$R = \{r(i,j) \mid 1 \le i \le m,\ 1 \le j \le n\} \cup \{b_1, \ldots, b_s\}.$$

Let $b(x) = \prod_{k=1}^{s}(x - b_k)$. For $1 \le i \le m$ let $f_i(x) = \prod_{k=1}^{n}(x - r(i,k))$ and for $1 \le j \le n$ let $g_j(x) = \prod_{k=1}^{m}(x - r(k,j))$.

By Lemma 6, we construct monic irreducible polynomials $F_i, G_j \in \mathbb{Z}[x]$, pairwise non-associated in $\mathbb{Q}[x]$, such that the product of any selection of the polynomials $(x - b_1), \ldots, (x - b_s), f_1, \ldots, f_m, g_1, \ldots, g_n$ has the same fixed divisor as the corresponding product in which $f_i$ has been replaced by $F_i$ and $g_j$ by $G_j$ for $1 \le i \le m$ and $1 \le j \le n$. Let

$$H(x) = \frac{1}{p} b(x) \prod_{i=1}^{m} F_i(x) \prod_{j=1}^{n} G_j(x),$$

then, by Lemma 4, $H$ has a factorization into $m+1$ irreducibles

$$H(x) = F_1(x) \cdot \ldots \cdot F_m(x) \cdot \frac{b(x) G_1(x) \cdot \ldots \cdot G_n(x)}{p}$$

and an essentially different factorization into $n+1$ irreducibles

$$H(x) = \frac{b(x) F_1(x) \cdot \ldots \cdot F_m(x)}{p} \cdot G_1(x) \cdot \ldots \cdot G_n(x)$$

and no other essentially different factorization. $\square$

**Corollary.** (Chapman and McClain [3]). $\mathrm{Int}(\mathbb{Z})$ *is fully elastic.*

**Theorem 9.** *Given natural numbers $1 \le m_1 \le \ldots \le m_n$, we can construct a polynomial $H \in \mathrm{Int}(\mathbb{Z})$ that has exactly $n$ essentially different factorizations into irreducibles in $\mathrm{Int}(\mathbb{Z})$, the lengths of these factorizations being $m_1 + 1, \ldots, m_n + 1$.*

*Proof.* Let $N = (\sum_{i=1}^{n} m_i)^2 - \sum_{i=1}^{n} m_i^2$, and $p > N$ prime, $s = p - N$. By Lemma 5, we construct a complete system of residues $R$ mod $p$ that does not contain a complete system of residues for any prime $q < p$. We partition $R$ into disjoint sets $R = R_0 \cup \{t_1, \ldots, t_s\}$ with $|R_0| = N$. The elements of $R_0$ are indexed as follows:

$$R_0 = \{r(k,h,i,j) \mid 1 \le k \le n,\ 1 \le h \le m_k,\ 1 \le i \le n,\ 1 \le j \le m_i;\ i \ne k\},$$

meaning we arrange the elements of $R_0$ in an $m \times m$ matrix with $m = m_1 + \ldots + m_n$, whose rows and columns are partitioned into $n$ blocks of sizes $m_1, \ldots, m_n$. Now $r(k,h,i,j)$ designates the entry in the $h$-th row of the $k$-th block of rows and the $j$-th column of the $i$-th block of columns. Positions in the matrix whose row and column are each in block $i$ are left empty: there are no elements $r(k,h,i,j)$ with $i = k$.

For $1 \le k \le n$, $1 \le h \le m_k$, let $S_{k,h}$ be the set of entries in the $(k,h)$-th row:

$$S_{k,h} = \{r(k,h,i,j) \mid 1 \le i \le n,\ i \ne k,\ 1 \le j \le m_i\}.$$

For $1 \le i \le n$, $1 \le j \le m_i$, let $T_{i,j}$ be the set of elements in the $(i,j)$-th column:

$$T_{i,j} = \{r(k,h,i,j) \mid 1 \le k \le n,\ k \ne i,\ 1 \le h \le m_k\}.$$

For $1 \le k \le n$, $1 \le h \le m_k$, set

$$f_h^{(k)}(x) = \prod_{r \in S_{k,h}} (x - r) \cdot \prod_{r \in T_{k,h}} (x - r).$$

Also, let $b(x) = \prod_{i=1}^{s} (x - t_i)$.

By Lemma 6, we construct monic irreducible polynomials $F_h^{(k)}$, pairwise non-associated in $\mathbb{Q}[x]$, with $\deg F_h^{(k)} = \deg f_h^{(k)}$, such that any product of a selection of polynomials from $(x - t_1), \ldots, (x - t_s)$ and $f_h^{(k)}$ for $1 \le k \le n$, $1 \le h \le m_k$ has the same fixed divisor as the corresponding product in which the $f_h^{(k)}$ have been replaced by the $F_h^{(k)}$. Let

$$H(x) = \frac{1}{p} b(x) \prod_{k=1}^{n} \prod_{h=1}^{m_k} F_h^{(k)}(x).$$

Then $\deg H = N + p$; and for each $i = 1, \ldots, n$, $H$ has a factorization into $m_i + 1$ irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$:

$$H(x) = F_1^{(i)}(x) \cdot \ldots \cdot F_{m_i}^{(i)}(x) \cdot \frac{b(x) \prod_{k \ne i} \prod_{h=1}^{m_k} F_h^{(k)}(x)}{p}$$

These factorizations are essentially different, since the $F_j^{(i)}$ are pairwise non-associated in $\mathbb{Q}[x]$ and hence in $\mathrm{Int}(\mathbb{Z})$.

By Lemma 4, $H$ has no further essentially different factorizations. This is so because a minimal subset with fixed divisor $p$ of the polynomials $(x-t_i)$ for $1 \le i \le s$ and $F_h^{(k)}$ for $1 \le k \le n$, $1 \le h \le m_k$ must consist of all the linear factors $(x-t_i)$ together with a minimal selection of $F_h^{(k)}$ such that all $r \in R_0$ occur as roots in the product of the corresponding $f_h^{(k)}$. For all linear factors $(x-r)$ with $r \in R_0$ to occur in a set of polynomials $f_h^{(k)}$, it must contain for all but one $k$ all $f_h^{(k)}$, $h = 1, \ldots m_k$. If, for $i \neq k$, $f_h^{(k)}$ and $f_j^{(i)}$ are missing, then $r(k,h,i,j)$ and $r(i,j,k,h)$ do not occur among the roots of the polynomials $f_h^{(k)}$. A set consisting of all $f_h^{(k)}$ for $n-1$ different values of $k$, however, has the property that all linear factors $(x-r)$ for $r \in R_0$ occur. □

**Corollary.** *Every finite subset of* $\mathbb{N}\setminus\{1\}$ *occurs as the set of lengths of a polynomial* $f \in \mathrm{Int}(\mathbb{Z})$.

## 5. *No transfer homomorphism to a block-monoid*

For some monoids, results like the above Corollary have been shown by means of transfer-homomorphisms to block monoids. For instance, by Kainrath [6], in the case of a Krull monoid with infinite class group such that every divisor class contains a prime divisor.

$\mathrm{Int}(\mathbb{Z})$, however, doesn't admit this method: We will show a property of the multiplicative monoid of $\mathrm{Int}(\mathbb{Z}) \setminus \{0\}$ that excludes the existence of a transfer-homomorphism to a block monoid.

**Proposition 10.** *For every* $n \ge 1$ *there exist irreducible elements* $H, G_1, \ldots, G_{n+1}$ *in* $\mathrm{Int}(\mathbb{Z})$ *such that* $xH(x) = G_1(x) \ldots G_{n+1}(x)$.

*Proof.* Let $p_1 < p_2 < \ldots < p_n$ be $n$ distinct odd primes, $P = \{p_1, p_2, \ldots, p_n\}$, and $Q$ the set of all primes $q \le p_n + n$. By the Chinese remainder theorem construct $a_1, \ldots, a_n$ with $a_i \equiv 0 \mod p_i$ and $a_i \equiv 1 \mod q$ for all $q \in Q$ with $q \neq p_i$. Similarly, construct $b_1, \ldots b_{p_n}$ such that, firstly, for all $p \in P$, $b_k \equiv k \mod p$ if $k \le p$ and $b_k \equiv 1 \mod p$ if $k > p$ and, secondly, $b_k \equiv 1 \mod q$ for all $q \in Q \setminus P$. So, for each $p_i \in P$, a complete set of residues mod $p_i$ is given by $b_1, \ldots b_{p_i}, a_i$, while all remaining $a_j$ and $b_k$ are congruent to 1 mod $p_i$. Also, all $a_j$ and $b_k$ are congruent to 1 for all primes in $Q \setminus P$.

Set $f(x) = (x-b_1) \ldots (x-b_{p_n})$ and let $F(x)$ be a monic irreducible polynomial in $\mathbb{Z}[x]$ with $\deg F = \deg f$ such that the fixed divisor of any product of a selection of polynomials from $f(x), (x-a_1), \ldots, (x-a_n)$ is the same as the fixed divisor of the corresponding set of polynomials in which $f$ has been replaced by $F$. Such an $F$ exists by Lemma 6. Let

$$H(x) = \frac{F(x)(x-a_1)\ldots(x-a_n)}{p_1\cdots p_n}.$$

Then $H(x)$ is irreducible in $\mathrm{Int}(\mathbb{Z})$, and

$$xH(x) = \frac{xF(x)}{p_1\cdots p_n}\cdot(x-a_1)\cdot\ldots\cdot(x-a_n),$$

where $xF(x)/(p_1\ldots p_n)$ and, of course, $(x-a_1)$, ..., $(x-a_n)$, are irreducible in $\mathrm{Int}(\mathbb{Z})$. $\square$

**Remark 11.** *Thanks to Roger Wiegand for suggesting an easier proof of Proposition 10: Using the well-known fact that the binomial polynomials* $\binom{x}{m}$ *are irreducible in* $\mathrm{Int}(\mathbb{Z})$ *for all* $m>0$*, it suffices to consider*

$$x\binom{x-1}{m-1} = m\binom{x}{m}$$

*with* $m$ *chosen to have exactly* $n$ *prime factors in* $\mathbb{Z}$*.*

**Remark 12.** *Thanks to Alfred Geroldinger for pointing this out: Proposition 10 implies that there does not exist a transfer-homomorphism from the multiplicative monoid* $(\mathrm{Int}(\mathbb{Z})\setminus\{0\},\cdot)$ *to a block-monoid. (For the definition of block-monoid and transfer-homomorphism see* [5] *Def. 2.5.5 and Def. 3.2.1, respectively.)*

*This is so because, in a block-monoid, the length of factorizations of elements of the form* $cd$ *with* $c$*,* $d$ *irreducible,* $c$ *fixed, is bounded by a constant depending only on* $c$*, cf.* [5]*, Lemma 6.4.4. More generally, applying* [5]*, Lemma 3.2.2, one sees that every monoid that admits a transfer-homomorphism to a block-monoid has this property, in marked contrast to Proposition 10.*

## References


[1] P.-J. Cahen and J.-L. Chabert, *Elasticity for integral-valued polynomials*, J. Pure Appl. Algebra 103 (1995), 303–311.

[2] P.-J. Cahen and J.-L. Chabert, *Integer-valued polynomials*, vol. 48 of Mathematical Surveys and Monographs, Amer. Math. Soc., 1997.

[3] S. T. Chapman and B. A. McClain, *Irreducible polynomials and full elasticity in rings of integer-valued polynomials*, J. Algebra 293 (2005), 595–610.

[4] Ch. Frei and S. Frisch, *Non-unique factorization of polynomials over residue class rings of the integers*, Comm. Algebra 39 (2011), 1482–1490.

[5] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations*, vol. 278 of Pure and Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 2006.

[6] F. Kainrath, *Factorization in Krull monoids with infinite class group*, Colloq. Math. 80 (1999), 23–30.

Institut für Mathematik A
Technische Universität Graz
Steyrergasse 30
A-8010 Graz, Austria
frisch@tugraz.at