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Substitution and Closure of Sets under Integer-Valued Polynomials

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Abstract.

Let R be a domain and K its quotient-field. For a subset S of K, let $\mathcal{F}_R(S)$ be the set of polynomials $f \in K[x]$ with $f(S) \subseteq R$ and define the R-closure of S as the set of those $t \in K$ for which $f(t) \in R$ for all $f \in \mathcal{F}_R(S)$. The concept of R-closure was introduced by McQuillan (*J. Number Theory* **39** (1991), 245–250), who gave a description in terms of closure in P-adic topology, when R is a Dedekind ring with finite residue fields. We introduce a toplogy related to, but weaker than P-adic topology, which allows us to treat ideals of infinite index, and derive a characterization of R-closure when R is a Krull ring. This gives us a criterion for $\mathcal{F}_R(S) = \mathcal{F}_R(T)$, where S and T are subsets of K. As a corollary we get a generalization to Krull rings of R. Gilmer's result (*J. Number Theory* **33** (1989), 95–100) characterizing those subsets S of a Dedekind ring with finite residue fields for which $\mathcal{F}_R(S) = \mathcal{F}_R(R)$.

1. Introduction.

Let R be a domain and K its quotient-field. The ring of integer-valued polynomials on R consists of those polynomials in $K[x]$ that map R to itself, when acting as a function on K by substitution of the variable. (The name stems from the classical case, where R is the ring of integers in a number field.) Although this ring has been the object of extensive study (originating with two seminal papers by Pólya [5] and Ostrowski [4]), some natural questions have not been considered until fairly recently. If, for a subset S of K, we denote by $\mathcal{F}_R(S)$ the set of R-valued polynomials on S, $\mathcal{F}_R(S) = \{f \in K[x] \mid f(S) \subseteq R\}$, a question one may ask is which subsets of R can be substituted for R to define the ring of integer-valued polynomials, $\mathcal{F}_R(S) = \mathcal{F}_R(R)$. R. Gilmer [1] characterized those subsets for a Dedekind ring with finite residue fields.

To investigate when $\mathcal{F}_R(S) = \mathcal{F}_R(T)$, for arbitrary $S, T \subseteq K$, D. L. McQuillan [3] introduced the R-closure of a set, $R\text{–cl}(S) = \{t \in K \mid \forall f \in \mathcal{F}_R(S)\ f(t) \in R\}$. Clearly, $\mathcal{F}_R(S) \subseteq \mathcal{F}_R(T)$ if and only if $T \subseteq R\text{–cl}(S)$. For a Dedekind ring with finite residue fields McQuillan gave a description of the R-closure in terms of the closures in P-adic topology, where P runs through the maximal ideals of R.

In this paper we introduce a topology related to, but weaker than P-adic topology, which allows us to handle prime ideals of infinite index. When R is a Dedekind ring (or more generally a Krull ring), we give a characterization of the R-closure of sets in terms of "weak P-adic topology." As a corollary we get a generalization of Gilmer's result to Krull rings.

2. Weak $\mathcal{I}$-adic topology.

All rings considered will be commutative with identity. A descending chain of ideals in a ring R is understood to be a sequence $\mathcal{I} = \{I_n \mid n \in \mathbb{N}\}$ of ideals with $I_{n+1} \subseteq I_n$ and we set $I_0 := R$. (The natural numbers $\mathbb{N}$ do not contain 0, but $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.)

Definition. Let R be a ring, $\mathcal{I}$ a descending chain of ideals in R and M an R-module. We define *weak $\mathcal{I}$-adic topology on M* by giving a neighborhood basis for $m \in M$: $\mathcal{U}(m) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(m)$, where $\mathcal{U}_n(m)$ consists of all sets $M \setminus \bigcup_{j=1}^{n} E_j$, such that each E_j is contained in $m + I_{j-1}M$ and is a finite union of residue classes of I_jM, other than $m + I_jM$, in M. (Thus $\mathcal{U}_n(m) = \{m + U \mid U \in \mathcal{U}_n(0)\}$.)

Definition. If I is an ideal of R, *weak I-adic topology* is defined as weak $\mathcal{I}$-adic topology for $\mathcal{I} = \{I^n \mid n \in \mathbb{N}\}$.

To see that the neighborhood bases $\mathcal{U}(m)$ define a topology on M we check that
(1) $\forall\, U \in \mathcal{U}_n(m)\;\; m + I_nM \subseteq U$, in particular, $m \in U$,
(2) $U, V \in \mathcal{U}_n(m) \implies U \cap V \in \mathcal{U}_n(m)$,
(3) $\forall z \in U \in \mathcal{U}_n(m)\;\; \exists V \in \mathcal{U}_n(z)\;\; V \subseteq U$.

Ad (3): If $z \equiv m \mod I_nM$ then $\mathcal{U}_n(z) = \mathcal{U}_n(m)$. If $l < n$ is maximal such that $z \equiv m \mod I_lM$ and $U = M \setminus \bigcup_{j=1}^{n} E_j$, each E_j being a finite union of residue classes other than $m + I_jM$ in $m + I_{j-1}M$, then $V = M \setminus \left(\bigcup_{j=1}^{l+1} E_j \,\cup\, (m + I_{l+1}M)\right) \subseteq U$ and $V \in \mathcal{U}_{l+1}(z) \subseteq \mathcal{U}_n(z)$.

Remarks. (3) shows that basis neighborhoods are open and (1) implies that weak I-adic topology is actually weaker than I-adic topology.

Perhaps a more natural way to look at weak $\mathcal{I}$-adic topology on a ring R is the following: If $\bigcap_{n=1}^{\infty} I_n = (0)$ then there is an embedding ι of R (otherwise of $R \,/\, \bigcap_{n=1}^{\infty} I_n$) into $\prod_{n=1}^{\infty} I_{n-1}/I_n$ (for $n \in \mathbb{N}$ let $\{{c_j}^{(n)} \mid 0 \leq j < [I_{n-1} : I_n]\}$ be a residue system of I_{n-1} mod I_n and for $r \in R$ define $\iota(r) := \left({c_{j_n(r)}}^{(n)}\right)_{n=1}^{\infty}$ by $r \equiv \sum_{n=1}^{N} {c_{j_n(r)}}^{(n)} \mod I_N$ for all $N \in \mathbb{N}$). Weak $\mathcal{I}$-adic topology is then induced on R by the product topology of co-finite topology on each factor I_{n-1}/I_n. If, for an ideal I in R, we compare I-adic topology to weak I-adic topology, we see that the former is induced by product topology of discrete topology on $\prod_{n=1}^{\infty} I^{n-1}/I^n$, and thus is stronger than the latter, and that equality holds if and only if $[I^{n-1} : I^n]$ is finite for all $n \in \mathbb{N}$.

3. Local investigations.

Throughout the "local" section, v is a discrete valuation (with value group equal to $\mathbb{Z}$ and $v(0) = \infty$) on a field K and R_v its valuation ring with maximal ideal M_v. If S is a set contained in R_v we denote the closure of S in weak M_v-adic topology by $\overline{S}$. We shall see that weak M_v-adic topology on R_v arises naturally as "topology of closure under integer-valued polynomials," in that $\overline{S} = R_v\text{--cl}(S)$. We need a few technical Lemmata.

LEMMA 1. *Let R be a subring of a ring R', and $\mathcal{I}$, $\mathcal{J}$ descending chains of ideals in R and R', respectively. If there exists a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ with $\varphi(1) = 1$ such that for all $n \in \mathbb{N}$, $I_n = J_k \cap R$ whenever $\varphi(n) \leq k < \varphi(n+1)$, then weak $\mathcal{I}$-adic topology on R is equal to the topology inherited from weak $\mathcal{J}$-adic topology .*

Proof. Fix $t \in R$. If C is a residue class of J_k in R' then either $R \cap C = \emptyset$ or $C = r + J_k$ and $C \cap R = r + (J_k \cap R)$ for some $r \in R$. Moreover, if $C = r + J_k$ with $r \in R$ such that $r \equiv t\ (J_{k-1})$, but $r \not\equiv t\ (J_k)$, then we must have $J_{k-1} \cap R \neq J_k \cap R$, so there exists $n \in \mathbb{N}$ with $k = \varphi(n)$; and if we put $D = C \cap R$ then $D = r + I_n \neq t + I_n$ and $D \subseteq t + I_{n-1}$. Conversely, if for some $r \in R$, $D = r + I_n$ with $r \equiv t\ (I_{n-1})$ and $r \not\equiv t\ (I_n)$ then $D = R \cap C$, where $C = r + J_{\varphi(n)} \subseteq t + J_{\varphi(n)-1}$ and $C \neq t + J_{\varphi(n)}$. It follows immediately from these considerations that the intersections of weak $\mathcal{J}$-adic basis neighborhoods of t with R are precisely the weak $\mathcal{I}$-adic basis neighborhoods of t. $\square$

Remark: If v' is an extension of the discrete valuation v to a finite-dimensional extension K' of K then Lemma 1 implies equality of weak M_v-adic topology on R_v with the topology inherited from weak $M_{v'}$-adic topology. Namely, if $e \in \mathbb{N}$ is the index of the valuation group of v in the valuation group of v' then ${M_v}^n = {M_{v'}}^k \cap R$ whenever $e \cdot (n-1) + 1 \leq k < e \cdot n + 1$.

LEMMA 2. *Let $f \in R_v[x]$, not all of whose coefficients lie in M_v, split over K, as $f(x) = d(x - b_1) \cdot \ldots \cdot (x - b_m) \cdot (x - c_1) \cdot \ldots \cdot (x - c_l)$, where $v(b_i) < 0$ and $v(c_i) \geq 0$, and put $f_+(x) = (x - c_1) \cdot \ldots \cdot (x - c_l)$. Then $v(f(r)) = v(f_+(r))$ for all $r \in R_v$.*

Proof. $\forall r \in R_v$ $v(r - b_i) = v(b_i)$ and $v(f(r)) = v(d) + \sum_{i=1}^{m} v(b_i) + v(f_+(r))$; we show $v(d) = -\sum_{j=1}^{m} v(b_i)$. Consider $d^{-1}f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Since $f \in R_v[x] \setminus M_v[x]$, $v(d) = -\min_{0 \leq i \leq n} v(a_i)$. But the a_i are the elementary symmetric polynomials in the b_i and c_i, so $\min_{0 \leq i \leq n} v(a_i) = v(a_{n-m}) = \sum_{i=1}^{m} v(b_i)$. $\square$

Lemma 3. *Let S be a set contained in R_v and $a \in R_v$. Then*

$$a \in \overline{S} \quad \Longrightarrow \quad \forall f \in K[x] \ \exists s \in S \ \ v(f(s)) \leq v(f(a)).$$

Proof. If $S = \emptyset$ or f is constant or $f(a) = 0$ the statement is trivial; from now on, assume $S \neq \emptyset$, $\deg(f) \geq 1$, and $f(a) \neq 0$. First consider a monic $f \in R_v[x]$ that splits over K: $f(x) = \prod_{i=1}^{n}(x - c_i)$ with $c_i \in R_v$. Since $f(a) \neq 0$, $l = \max_i v(a - c_i)$ exists, and $v(f(a)) = \sum_{i=1}^{n} v(a - c_i) = \sum_{j \geq 1} \left|\{i \mid a \equiv c_i \bmod M_v^j\}\right| = \sum_{j=1}^{l} \left|\{i \mid a \equiv c_i \bmod M_v^j\}\right|$.

Since $a \in \overline{S}$, and S therefore intersects every $U \in \mathcal{U}_{l+1}(a)$, either there exists $s_0 \in S \cap (a + M_v^{l+1})$, or there exists $m \leq l$ such that S intersects infinitely many residue classes of M_v^{m+1} in $a + M_v^m$. In the first case, $v(f(s_0)) = \sum_{j=1}^{l}\left|\{i \mid s_0 \equiv c_i \bmod M_v^j\}\right| = \sum_{j=1}^{l}\left|\{i \mid a \equiv c_i \bmod M_v^j\}\right| = v(f(a))$. In the second case, pick $t_0 \in S \cap (a + M_v^m)$ such that $t_0 \not\equiv c_i \bmod M_v^{m+1}$ for $i = 1, \ldots, n$ then $v(f(t_0)) = \sum_{j=1}^{m}\left|\{i \mid t_0 \equiv c_i \bmod M_v^j\}\right| = \sum_{j=1}^{m}\left|\{i \mid a \equiv c_i \bmod M_v^j\}\right| \leq v(f(a))$.

Now for a general $f \in K[x]$ (with $\deg(f) \geq 1$ and $f(a) \neq 0$), write f as $c \cdot g$ with $c \in K$, $g \in R_v[x] \setminus M_v[x]$. It suffices to prove the claim for g. Let K' be the splitting field of g over K, v' an extension of v to K' (normalized to have value group $\mathbb{Z}$, such that on K, we have $v' = e \cdot v$, $e \in \mathbb{N}$). Over K' we get $g(x) = d(x - c_1)\ldots(x - c_n)(x - b_1)\ldots(x - b_m)$ with $v'(c_i) \geq 0$, $v'(b_i) < 0$. By Lemma 2, for every $t \in R_{v'}$, $v'(g(t)) = v'(g_+(t))$, where $g_+(x) = (x - c_1)\ldots(x - c_n)$. But now we know there exists $s \in S$ with $v'(g_+(s)) \leq v'(g_+(a))$ (using the fact that the closure of S in weak M_v-adic topology is contained in the closure with respect to weak $M_{v'}$-adic topology); and $v(g(s)) = e^{-1}v'(g(s)) = e^{-1}v'(g_+(s)) \leq e^{-1}v'(g_+(a)) = e^{-1}v'(g(a)) = v(g(a))$. $\square$

Lemma 4. *Let S be a set contained in R_v and $a \in R_v$. Then*

$$a \notin \overline{S} \quad \Longrightarrow \quad \exists f \in [S][x] \ \forall s \in S \ \ v(f(s)) > v(f(a)),$$

where $[S]$ denotes the ring generated by S in K.

Proof. If $S = \emptyset$ the statement is trivial, so assume $S \neq \emptyset$. Since $a \notin \overline{S}$, there exists a basis-neighborhood of a which S doesn't intersect, and hence a minimal $N \in \mathbb{N}$ such that $S \cap (a + {M_v}^N) = \emptyset$ and S meets only finitely many residue classes of ${M_v}^n$ in $a + {M_v}^{n-1}$ for all $n \leq N$. Inductively, from $k = N - 1$ down to $k = 0$, we construct a sequence of polynomials $f_k \in [S][x]$ such that $v(f_k(s)) > v(f_k(a))$ for all $s \in S \cap (a + {M_v}^k)$.

Define $f_{N-1}(x) = \prod_{i=1}^{m}(x - s_i)$, where $s_1, \ldots, s_m \in S$ are representatives of the different residue classes of ${M_v}^N$ that S intersects in $a + {M_v}^{N-1}$. (Minimality of N and the fact that $S \neq \emptyset$ guarantee that S intersects $a + {M_v}^{N-1}$; hence $m \neq 0$.) Then $v(f_{N-1}(s)) \geq m(N-1) + 1 > m(N-1) = v(f_{N-1}(a))$ for all $s \in S \cap (a + {M_v}^{N-1})$.

Given f_k such that for all $s \in S \cap (a + {M_v}^k)$ $v(f_k(s)) \geq c$ while $v(f_k(a)) = c - 1$, we construct f_{k-1}. Set $d = \min\{v(f_k(s)) \mid s \in S \cap (a + {M_v}^{k-1})\}$. If $d \geq c$ then $f_{k-1} = f_k$ works. If $d < c$, let $t_1, \ldots, t_l \in S$ be representatives of the different residue classes of ${M_v}^k$ in $a + {M_v}^{k-1}$, other than $a + {M_v}^k$, that S intersects. Define $g(x) = \prod_{i=1}^{l}(x - t_i)$ and $f_{k-1} = g^{c-d} \cdot f_k$. Putting together the facts that

$$\forall s \in S \cap \left((a + {M_v}^{k-1}) \setminus (a + {M_v}^k)\right) \quad v(g(s)) \geq l(k-1) + 1 \quad \text{and} \quad v(f_k(s)) \geq d\,,$$

$$\forall t \in a + {M_v}^k \quad v(g(t)) = l(k-1)\,,$$

$$\text{and} \qquad \forall s \in S \cap (a + {M_v}^k) \quad v(f_k(s)) \geq c, \qquad \text{while} \quad v(f_k(a)) = c - 1\,,$$

we see that $v(f_{k-1}(a)) = (c-d)l(k-1) + c - 1$, while $v(f_{k-1}(s)) \geq (c-d)l(k-1) + c$ for all $s \in S \cap a + {M_v}^{k-1}$. $\square$

PROPOSITION 1. *If A and S are sets contained in R_v then*

$$\mathcal{F}_{R_v}(S) \subseteq \mathcal{F}_{R_v}(A) \qquad \Longleftrightarrow \qquad A \subseteq \overline{S}\,.$$

Proof. For any $a \in \overline{S}$ Lemma 3 shows that $\mathcal{F}_{R_v}(S) \subseteq \mathcal{F}_{R_v}(\{a\})$. Conversely, if $a \notin \overline{S}$, Lemma 4 allows us to construct a member of $\mathcal{F}_{R_v}(S) \setminus \mathcal{F}_{R_v}(\{a\})$ by multiplying the f in the Lemma by a constant $c \in K$ with $v(c) = -\min_{s \in S} v(f(s))$. The statement for A now follows from the fact that $\mathcal{F}_{R_v}(A) = \bigcap_{a \in A} \mathcal{F}_{R_v}(\{a\})$. $\square$

COROLLARY. *If A and S are sets contained in R_v then*

(i) $$\mathcal{F}_{R_v}(S) = \mathcal{F}_{R_v}(A) \qquad \Longleftrightarrow \qquad \overline{A} = \overline{S}$$

(ii) $$R_v\text{–cl}(S) = \overline{S}\,.$$

4. Results for Krull-rings.

From now on, let R be a Krull ring, K its field of fractions, and $\mathcal{P}$ the set of height 1 prime ideals of R. If $P \in \mathcal{P}$, we denote by $P^{(n)}$, $n \in \mathbb{N}$, the symbolic powers of P, $P^{(n)} = (P_P)^n \cap R$, where P_P is the extension of P to the localization R_P. By $\overline{S}$ we now mean the closure of S in the specified topology, be it weak $\{P^{(n)} \mid n \in \mathbb{N}\}$-adic, weak P-adic or P-adic. A subset S of K is called R-fractional if $S \subseteq d^{-1}R$ for some $d \in R$. As with Dedekind rings with finite residue fields (McQuillan [3]), the case of non-R-fractional sets is simple (I thank F. Halter-Koch for spiffying up the following proposition, which I had only shown for Krull rings, and in a more pedestrian manner.).

Proposition 2. *Let R be an integrally closed domain with quotient field K. If $A \subseteq K$ is not R-fractional then $\mathcal{F}_R(A)$ consists only of the constant polynomials with values in R and hence $R-\mathrm{cl}(A) = K$.*

Proof. Suppose $f \in \mathcal{F}_R(A)$, $\deg f = n > 0$. There exists $c \neq 0$ in R such that $cf = g \in R[x]$, $g(x) = c_n x^n + \ldots + c_0$, $c_n \neq 0$. For every $a \in A$, $g(a) \in R$ implies that $c_n a$ is integral over R, and therefore $c_n a \in R$. Thus $A \subseteq {c_n}^{-1}R$. □

We now turn to R-fractional sets.

Theorem 1. *Let A and B be subsets of $d^{-1}R$, $d \in R$, then*

(i) $\mathcal{F}_R(A) \subseteq \mathcal{F}_R(B) \iff \forall P \in \mathcal{P}\ \ B \subseteq \overline{A}$ *in weak* $\{P^{(n)}\}$*−adic topology on* $d^{-1}R$

(ii) $R-\mathrm{cl}(A)$ *is the intersection of all weak* $\{P^{(n)}\}$*− adic closures of* A, $P \in \mathcal{P}$.

Proof. In the case where $A, B \subseteq R$, we show that the following are equivalent:

$$\mathcal{F}_R(A) \subseteq \mathcal{F}_R(B) \tag{1}$$

$$\forall P \in \mathcal{P},\ \ B \subseteq \overline{A} \text{ in weak } P_P\text{−adic topology on } R_P \tag{2}$$

$$\forall P \in \mathcal{P},\ \ B \subseteq \overline{A} \text{ in weak } \{P^{(n)}\}\text{−adic topology on } R. \tag{3}$$

($1 \Rightarrow 2$) Suppose $B \not\subseteq \overline{A}$ in weak P_P-adic topology for some fixed $P \in \mathcal{P}$. Then by Lemma 4 there exists a polynomial $f \in [A][x]$ and an integer n, such that for all $a \in A$ $v_P(f(a)) \geq n$, and for some $b \in B$ $v_P(f(b)) < n$. By the Approximation Theorem for Krull-rings [2, p90], there is a $c \in K$ with $v_P(c) = -n$ and $v_Q(c) \geq 0$ for all $Q \neq P$, $Q \in \mathcal{P}$. Then $c \cdot f \in \mathcal{F}_{R_P}(A)$, but $c \cdot f \notin \mathcal{F}_{R_P}(B)$. Also, for $Q \neq P$, $Q \in \mathcal{P}$, $c \cdot f \in R_Q[x] \subseteq \mathcal{F}_{R_Q}(A)$. Therefore, $c \cdot f$ is in $\mathcal{F}_R(A)$, but not in $\mathcal{F}_R(B)$.

($2 \Rightarrow 1$) By Proposition 1, $B \subseteq \overline{A}$ in weak P_P-adic topology implies $\mathcal{F}_{R_P}(A) \subseteq \mathcal{F}_{R_P}(B)$. Using $R = \bigcap_{P \in \mathcal{P}} R_P$ we get $\mathcal{F}_R(A) = \bigcap_{P \in \mathcal{P}} \mathcal{F}_{R_P}(A) \subseteq \bigcap_{P \in \mathcal{P}} \mathcal{F}_{R_P}(B) = \mathcal{F}_R(B)$.

($2 \Leftrightarrow 3$) Weak $\{P^{(n)}\}$-adic topology on R is – by definition of $P^{(n)}$ and Lemma 1 – exactly what R inherits from weak P_P-adic topology on R_P.

To reduce the fractional sets case to the subsets of R case we convince ourselves that:

(4) $\mathcal{F}_R(A) \subseteq \mathcal{F}_R(B)$ if and only if $\mathcal{F}_R(dA) \subseteq \mathcal{F}_R(dB)$ and

(5) For every $P \in \mathcal{P}$, $B \subseteq \overline{A}$ in weak $\{P^{(n)}\}$-adic topology on $d^{-1}R$ if and only if $dB \subseteq \overline{dA}$ in weak $\{P^{(n)}\}$-adic topology on R.

Ad (4) Consider $\varphi_d \colon K[x] \to K[x]$, $\varphi_d(f(x)) = f\left(d^{-1}x\right)$. Clearly, $\varphi_d\left(\mathcal{F}_R(S)\right) = \mathcal{F}_R(dS)$ for any set $S \subseteq K$. Because φ_d is a permutation of $K[x]$, $\varphi_d(S) \subseteq \varphi_d(T)$ if and only if $S \subseteq T$ for all $S, T \subseteq K$.

Ad (5) $\psi: d^{-1}R \to R$, $\psi(x) = dx$ (as an R-module isomorphism) is a homeomorphism between the $\mathcal{I}$-adic topologies on $d^{-1}R$ and R for any descending sequence of ideals $\mathcal{I}$.

The characterization of $R-\mathrm{cl}(A)$ is now an easy consequence of its definition as the unique largest set B with $\mathcal{F}_R(A) \subseteq \mathcal{F}_R(B)$. $\square$

In what follows, we use the fact that $P^{(n)} = P^n$ whenever P^n is a primary ideal. This is always the case if P is a maximal ideal, but also when P is a principal prime ideal in a unique factorization domain; so that in these cases, weak $\{P^{(n)}\}$-adic topology is just weak P-adic topology. Also note that weak I-adic topology is equal to I-adic topology whenever $[R : I^n]$ is finite for all n, such that for a height 1 prime ideal P of finite index in a Krull ring, weak $\{P^{(n)}\}$-adic topology is simply P-adic topology. In the case of a Dedekind ring with finite residue fields, the following result is due to McQuillan [3].

Corollary. *Let $(R, \mathcal{P})$ be a Dedekind ring and its set of maximal ideals or a UFD and its set of principal prime ideals. If A and B are subsets of $d^{-1}R$, $d \in R$, then*

(i) $\mathcal{F}_R(A) \subseteq \mathcal{F}_R(B) \iff \forall P \in \mathcal{P}\ \ B \subseteq \overline{A}$ in weak $P-$adic topology on $d^{-1}R$

(ii) $R-\mathrm{cl}(A)$ *is the intersection of all weak P-adic closures of A, $P \in \mathcal{P}$.*

Theorem 2. *Let S be a set contained in a subring A of a Krull ring R. Then $\mathcal{F}_R(S) = \mathcal{F}_R(A)$ if and only if for every height 1 prime ideal P of R*

(a) *for all $n \in \mathbb{N}$ with $[A : P^{(n)} \cap A]$ finite, S contains a complete system of residues of $P^{(n)} \cap A$ in A and*

(b) *for the minimal N (if such exists) with $[A : P^{(N)} \cap A]$ infinite, S intersects infinitely many residue classes of $P^{(N)} \cap A$ in every residue class of $P^{(N-1)} \cap A$ in A.*

Proof. The condition is clearly necessary and sufficient for S to intersect every weak $\{P^{(n)}\}$-adic neighborhood for all $P \in \mathcal{P}$ of every $a \in A$, that is for A to be contained in the closure of S in weak $\{P^{(n)}\}$-adic topology for all $P \in \mathcal{P}$. $\square$

Corollary 1. *If S is a subset of a Krull ring R then $\mathcal{F}_R(S) = \mathcal{F}_R(R)$ if and only if S contains a complete residue system of P^n in R for every $n \in \mathbb{N}$ for every finite index $P \in \mathcal{P}$ and infinitely many elements incongruent mod P for every $P \in \mathcal{P}$ of infinite index.*

Proof. Every finite index prime ideal P is maximal, therefore P^n is primary and hence $P^n = P^{(n)}$ for all n; and the only height 1 prime ideals P in a Krull ring with $[R : P^{(n)}]$ infinite for some n are those of infinite index. $\square$

Finally, when $A = R$ in the following statement, we retrieve Gilmer's [1] result.

Corollary 2. *If R is a Dedekind ring with finite residue fields, A a subring of R and $S \subseteq A$ then $\mathcal{F}_R(S) = \mathcal{F}_R(A)$ if and only if S contains a complete set of residues of $P^n \cap A$ in A for every prime ideal P of R and every $n \in \mathbb{N}$.*

References

1. R. Gilmer, Sets That Determine Integer-Valued Polynomials, *J. Number Theory* **33** (1989), 95–100.

2. H. Matsumura, "Commutative ring theory," Cambridge University Press, 1986.

3. D. L. McQuillan, On a Theorem of R. Gilmer, *J. Number Theory* **39** (1991), 245–250.

4. A. Ostrowski, Über ganzwertige Polynome in algebraischen Zahlkörpern, *J. reine angew. Mathematik,* **149** (1919) 117–124.

5. G. Pólya, Über ganzwertige Polynome in algebraischen Zahlkörpern, *J. reine angew. Mathematik,* **149** (1919) 97–116.

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