ON THE SPECTRUM OF RINGS OF FUNCTIONS

SOPHIE FRISCH

Abstract. For $D$ a domain and $E \subseteq D$, we investigate the prime spectrum of rings of functions from $E$ to $D$, that is, of rings contained in $\prod_{e \in E} D$ and containing $D$. Among other things, we characterize, when $M$ is a maximal ideal of finite index in $D$, those prime ideals lying above $M$ which contain the kernel of the canonical map to $\prod_{e \in E} (D/M)$ as being precisely the prime ideals corresponding to ultrafilters on $E$. We give a sufficient condition for when all primes above $M$ are of this form and thus establish a correspondence to the prime spectra of ultraproduc ts of residue class rings of $D$. As a corollary, we obtain a description using ultrafilters, differing from Chabert’s original one which uses elements of the $M$-adic completion, of the prime ideals in the ring of integer-valued polynomials $\text{Int}(D)$ lying above a maximal ideal of finite index.

1. Introduction

Let $D$ be an integral domain, $E \subseteq D$, and $R$ a subring of $\prod_{e \in E} D$, containing $D$. The elements of $R$ can be interpreted as functions from $E$ to $D$ and, consequently, we call $R$ a ring of functions from $E$ to $D$. We will investigate the prime spectra of such rings of functions. We obtain, for quite general $R$, a partial description of the prime spectrum, cf. Theorems 3.7 and 5.3, and in special cases a complete characterization, cf. Corollary 6.5.

Our motivation is the spectrum of a ring of integer-valued polynomials: For $D$ an integral domain with quotient field $K$, let $\text{Int}(D) = \{ f \in K[x] \mid f(D) \subseteq D \}$ be the ring of integer-valued polynomials on $D$. More generally, when $K$ is understood, we let $\text{Int}(A, B) = \{ f \in K[x] \mid f(A) \subseteq B \}$ for $A, B \subseteq K$.

If $D$ is a Noetherian one-dimensional domain, a celebrated theorem of Chabert [1, Ch. V] states that every prime ideal of $\text{Int}(D)$ lying over a maximal ideal $M$ of finite index in $D$ is maximal and of the form

$$M_\alpha = \{ f \in \text{Int}(D) \mid f(\alpha) \in \hat{M} \},$$

where $\alpha$ is an element of the $M$-adic completion $\hat{D}_M$ of $D$ and $\hat{M}$ the maximal ideal of $\hat{D}_M$.

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In fact, Chabert showed two separate statements independently – both under the assumption that $D$ is Noetherian and one-dimensional and $M$ a maximal ideal of finite index of $D$:

1. Every maximal ideal of $\text{Int}(D)$ containing $\text{Int}(D, M)$ is of the form $M_\alpha$ for some $\alpha \in \hat{D}_M$.
2. Every maximal ideal of $\text{Int}(D)$ lying over $M$ contains $\text{Int}(D, M)$.

For a simplified proof of Chabert’s result, see [4], Lemma 4.4 and the remark following it.

We will show that a modified version of statement (1) holds in far greater generality, for rings of functions. The modification consists in replacing elements of the $M$-adic completion by ultrafilters.

Whether (2) holds or not for a particular $D$ and a particular subring of $D^E$ will have to be examined separately. It is, in some sense, a question of density of the subring in the product $\prod_{e \in E} D$.

We will work in the following setting:

**Definition 1.1.** Let $D$ be a commutative ring and $E \subseteq D$. Let $R$ be a commutative ring and $\varphi : R \to \prod_{e \in E} D$ a monomorphism of rings. $\varphi$ allows us to interpret the elements of $R$ as functions from $E$ to $D$.

If all constant functions are contained in $\varphi(R)$, we call the pair $(R, \varphi)$ a ring of functions from $E$ to $D$. We use $R = \mathcal{R}(E, D)$ (where $\varphi$ is understood) to denote a ring of functions from $E$ to $D$.

**Remark 1.2.** For our considerations it is vital that $\mathcal{R} = \mathcal{R}(E, D)$ contain all constant functions, because we will make extensive use of the following fact: when $I$ is an ideal of $\mathcal{R} = \mathcal{R}(E, D)$, $f \in I$ and $g \in D[x]$ a polynomial with zero constant term, then $g(f) \in I$, and similarly, if $g$ is a polynomial in several variables over $D$ with zero constant term, and an element of $I$ is substituted for each variable in $g$, then, an element of $I$ results.

Let us note that considerable research has been done on the spectrum of a power of a ring $D^E = \prod_{d \in E} D$ or a product of rings $\prod_{e \in E} D_e$. Gilmer and Heinzer [5, Prop. 2.3] have determined the spectrum of an infinite product of local rings, and Levy, Loustauau and Shapiro [8] that of an infinite power of $\mathbb{Z}$. Our focus here is not on the full product of rings, but on comparatively small subrings and the question of how much information about the spectrum of a ring can be obtained from its embedding in a power of a domain.

One ring can be embedded in different products: $\text{Int}(D)$ can be seen as a ring of functions from $K$ to $K$ as well as a ring of functions from $D$ to $D$. We will glean a lot more information about the spectrum of $\text{Int}(D)$ from the second interpretation than from the first.
2. Prime ideals corresponding to ultrafilters

Let $R = R(E, D)$ be a ring of functions from $E$ to $D$ as in Definition 1.1. We will now make precise the concept of ideals corresponding to ultrafilters, and the connection to ultraproducts $\prod_{e \in E}^U (D/M)$, where $M$ is a maximal ideal of $D$, and $U$ an ultrafilter on $E$. First a quick review of filters, ultrafilters and ultraproducts:

Definition 2.1. Let $S$ be a set. A non-empty collection $F$ of subsets of $S$ is called a filter on $S$ if

1. $\emptyset \notin F$.
2. $A, B \in F$ implies $A \cap B \in F$.
3. $A \subseteq C \subseteq S$ with $A \in F$ implies $C \in F$.

A filter $F$ on $S$ is called an ultrafilter on $S$ if, for every $C \subseteq S$, either $C \in F$ or $S \setminus C \in F$.

Remark 2.2. Clearly, a necessary and sufficient condition for $C \subseteq \mathcal{P}(S)$ to be contained in a filter on $S$ is that $C$ satisfies the finite intersection property. If the finite intersection property is satisfied, then the supersets of finite intersections of members of $C$ form a filter.

Although, strictly speaking, we do not need ultraproducts to prove our results, we will nevertheless introduce them, because they provide context, in particular to Lemma 2.6, and to sections 3 and 5.

Definition 2.3. Let $S$ be an index set and $U$ an ultrafilter on $S$. Suppose we are given, for each $s \in S$, a ring $R_s$. Then the ultraproduct of rings $\prod_{s \in S}^U R_s$ is defined as the direct product $\prod_{s \in S} R_s$ modulo the congruence relation

$$(r_s)_{s \in S} \sim (t_s)_{s \in S} \iff \{s \in S \mid r_s = t_s\} \in U.$$

Ultraproducts of other algebraic structures are defined analogously. The usefulness of ultraproducts is captured by the Theorem of L"os (cf. [6, Chpt. 3.2] or [7, Prop 1.6.14]) which states that an ultraproduct $\prod_{s \in S}^U R_s$ satisfies a first-order formula if and only if the set of indices $s$ for which $R_s$ satisfies the formula is in $U$. Here first-order formula means a formula in the first-order language whose only non-logical symbols (apart from the equality sign) are symbols for the algebraic operations; for instance, $+$ and $\cdot$ in the case of an ultraproduct of rings.

Definition 2.4. Let $D$ be a domain, $E \subseteq D$, $R = R(E, D)$ a ring of functions, $I$ an ideal of $D$ and $\mathcal{F}$ a filter on $E$.

For $f \in R(E, D)$, we let $f^{-1}(I) = \{e \in E \mid f(e) \in I\}$ and define

$I_\mathcal{F} = \{f \in R(E, D) \mid f^{-1}(I) \in \mathcal{F}\}$
Remark 2.5. Let everything as in Definition 2.4, I, J ideals of D and F, G filters on E. Some easy consequences of Definition 2.4 are:

(1) If $I \neq D$ then $IF \neq R$.
(2) $IF$ is an ideal of $R$ containing $R(E, I) = \{ f \in R \mid f(E) \subseteq I \}$.
(3) $I \subseteq J \implies IF \subseteq JF$.
(4) $F \subseteq G \implies IF \subseteq IG$.

Lemma 2.6. Let $D$ be a domain, $E \subseteq D$, and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions from $E$ to $D$.

Then for every prime ideal $P$ of $D$ and every ultrafilter $\mathcal{U}$ on $E$, $P\mathcal{U}$ is a prime ideal of $\mathcal{R}$.

Proof. Easy direct verification: let $fg \in P\mathcal{U}$; because $P$ is a prime ideal of $D$, the inverse image of $P$ under $f \cdot g$ is the union of $f^{-1}(P)$ and $g^{-1}(P)$. If the union of two sets is in an ultrafilter, then one of them must be in the ultrafilter. Therefore, $f \in P\mathcal{U}$ or $g \in P\mathcal{U}$. Also, $P\mathcal{U}$ cannot be all of $\mathcal{R}$ because it doesn’t contain the constant function 1. \qed

One way of looking at $P\mathcal{U}$ is by considering the following commuting diagram of ring-homomorphisms, where $\pi$ and $\pi_1$ mean applying the canonical projection in each factor of the product, and $\sigma$ and $\sigma_1$ mean factoring through the defining congruence relation of an ultraproduct.

\[
\begin{array}{cccc}
\mathcal{R} & \xrightarrow{\varphi} & \prod_{e \in E} D & \xrightarrow{\sigma_1} \prod_{e \in E}^\mathcal{U} D \\
\downarrow{\pi} & & \downarrow{\pi_1} & \\
\prod_{e \in E} (D/P) & \xrightarrow{\sigma} & \prod_{e \in E}^\mathcal{U} (D/P)
\end{array}
\]

$P\mathcal{U}$ is the kernel of the following composition of ring homomorphisms:

$\varphi : \mathcal{R} \rightarrow \prod_{e \in E} D$

followed by the canonical projection

$\pi : \prod_{e \in E} D \rightarrow \prod_{e \in E} (D/P)$

and the canonical projection

$\sigma : \prod_{e \in E} (D/P) \rightarrow \prod_{e \in E}^\mathcal{U} (D/P)$

Since $D/P$ is an integral domain, any ultraproduct of copies of $D/P$ is also an integral domain, by the Theorem of Loś. Therefore (0) is a prime ideal of $\prod_{e \in E}^\mathcal{U} (D/P)$ and hence $P\mathcal{U}$ a prime ideal of $\mathcal{R}$. We also see that $P\mathcal{U}$ is the inverse
image of a prime ideal of $\prod_{e \in E} D$ under $\varphi$, and further, of a prime ideal of the ultraprodut $\prod_{e \in E} D$ under $\sigma_1 \circ \varphi$.

3. THE SET OF ZERO-LOCI MOD $M$ OF AN IDEAL OF THE RING OF FUNCTIONS

As before, $D$ is a domain with quotient field $K$, $E \subseteq D$ and $R = R(E, D)$ a ring of functions from $E$ to $D$ as in Def. 1.1. Especially, recall from Def. 1.1 that $R$ is assumed to contain all constant functions.

Definition 3.1. For $M \subseteq D$ and $f \in R = R(E, D)$, let

$$f^{-1}(M) = \{ e \in E \mid f(e) \in M \}.$$

For an ideal $M$ of $D$ and an ideal $I$ of $R$, let

$$Z_M(I) = \{ f^{-1}(M) \mid f \in I \}.$$

Recall from Def. 2.4 that for a filter $F$ on $E$,

$$M_F = \{ f \in R(E, D) \mid f^{-1}(M) \in F \}.$$

Remark 3.2. Note that the above definition implies

1. $I \subseteq J \implies Z_M(I) \subseteq Z_M(J)$
2. $I \subseteq M_F \iff Z_M(I) \subseteq F$

Lemma 3.3. Let $M$ be an ideal of $D$ and $I$ an ideal of $R$. The following are equivalent:

(a) There exists a filter $F$ on $E$ such that $I \subseteq M_F$.
(b) $Z_M(I)$ satisfies the finite intersection property.

Proof. If $I \subseteq M_F$, then $Z_M(I)$ is contained in $F$ and hence satisfies the finite intersection property. Conversely, if $Z_M(I)$ satisfies the finite intersection property then, by Remark 2.2, the supersets of finite intersections of sets in $Z_M(I)$ form a filter $F$ on $E$ for which $Z_M(I) \subseteq F$ and hence $I \subseteq M_F$. $\Box$

In the case where $R(E, D) = \prod_{e \in E} D$ is the ring of all functions from $E$ to $D$, much more can be said; see the papers by Gilmer and Heinzer [5, Prop. 2.3] (concerning local rings) and Levy, Loustaunau and Shapiro [8] (concerning $D = \mathbb{Z}$).

For a field $K$ that is not algebraically closed, we will need, for an arbitrary $n \geq 2$, an $n$-ary form that has no zero but the trivial one. For this purpose, recall how to define a norm form: if $L : K$ is an $n$-dimensional field extension, multiplication by any $w \in L$ is a $K$-endomorphism $\psi_w$ of $L$. For a fixed choice of a $K$-basis of $L$, map every $w \in L$ to the determinant of the matrix of $\psi_w$ with respect to the chosen basis. This mapping, regarded as a function of the coordinates of $w$ with respect to the chosen basis, is easily seen to be an $n$-ary form that has no zero but the trivial one.
Lemma 3.4. Let $M$ be a maximal ideal of $D$ such that $D/M$ is not algebraically closed. Then for every ideal $\mathcal{I}$ of $\mathcal{R} = \mathcal{R}(E, D)$, $\mathcal{Z}_M(\mathcal{I})$ is closed under finite intersections.

Proof. Given $f, g \in \mathcal{I}$, we show that there exists $h \in \mathcal{I}$ with

$$h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M).$$

Consider any finite-dimensional non-trivial field extension of $D/M$, and let $n$ be the degree of the extension. The norm form of this field extension is a homogeneous polynomial in $n \geq 2$ indeterminates whose only zero in $(D/M)^n$ is the trivial one. By identifying $n - 1$ variables, we get a binary form $\bar{s} \in (D/M)[x, y]$ with no zero in $(D/M)^2$ other than $(0, 0)$. Let $s \in D[x, y]$ be a binary form that reduces to $\bar{s}$ when the coefficients are taken mod $M$.

Now, given $f$ and $g$ in $\mathcal{I}$, we set $h = s(f, g)$. By the fact that $\mathcal{R}$ contains all constant functions, $h$ is in $\mathcal{I}$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. \qed

Lemma 3.5. Let $M$ be a maximal ideal of $D$ and $\mathcal{R} = \mathcal{R}(E, D)$ a ring of functions such that every $f \in \mathcal{R}$ takes values in only finitely many residue classes mod $M$.

Then for every ideal $\mathcal{I}$ of $\mathcal{R}$, $\mathcal{Z}_M(\mathcal{I})$ is closed under finite intersections.

Proof. Again, given $f, g \in \mathcal{I}$, we show that there exists $h \in \mathcal{I}$ with

$$h^{-1}(M) = f^{-1}(M) \cap g^{-1}(M).$$

Let $A, B \subseteq D/M$ be finite sets of residue classes of $D$ mod $M$ such that $f(E)$ is contained in the union of $A$ and $g(E)$ in the union of $B$.

We can interpolate any function from $(D/M)^2$ to $(D/M)$ at any finite set of arguments by a polynomial in $(D/M)[x, y]$. Pick $\bar{s} \in (D/M)[x, y]$ with $\bar{s}(0, 0) = 0$ and $\bar{s}(a, b) = 1$ for all $(a, b) \in (A \times B) \setminus \{(0, 0)\}$. Let $s \in D[x, y]$ be a polynomial with zero constant coefficient that reduces to $\bar{s}$ when the coefficients are taken mod $M$.

Now, given $f$ and $g$ in $\mathcal{I}$, we set $h = s(f, g)$. By the fact that $\mathcal{R}$ contains all constant functions, $h$ is in $\mathcal{I}$. Also, $h(e) \in M$ if and only if both $f(e) \in M$ and $g(e) \in M$, as desired. \qed

Definition 3.6. Let $\mathcal{R} = \mathcal{R}(E, D)$ be a ring of functions and $M$ an ideal of $D$. We call $f \in \mathcal{R}$ an $M$-unit-valued function if $f(e) + M$ is a unit in $D/M$ for every $e \in E$.

Theorem 3.7. Let $M$ be a maximal ideal of $D$ and $\mathcal{I}$ an ideal of $\mathcal{R} = \mathcal{R}(E, D)$. Assume that either $D/M$ is not algebraically closed or that each function in $\mathcal{R}$ takes values in only finitely many residue classes mod $M$.

1. $\mathcal{I}$ is contained in an ideal of the form $M_F$ for some filter $\mathcal{F}$ on $E$ if and only if $\mathcal{I}$ contains no $M$-unit-valued function.

2. Every ideal $\mathcal{Q}$ of $\mathcal{R}$ that is maximal with respect to not containing any $M$-unit-valued function is of the form $M_U$ for some ultrafilter $\mathcal{U}$ on $E$. 

In particular, every maximal ideal of \( \mathcal{R} \) that does not contain any \( \mathcal{M} \)-unit-valued function is of the form \( \mathcal{M}_U \) for some ultrafilter \( U \) on \( E \).

**Proof.** Ad (1). If \( \mathcal{I} \) is contained in an ideal of the form \( \mathcal{M}_F \), \( \mathcal{I} \) cannot contain any \( \mathcal{M} \)-unit-valued function, because \( \mathcal{F} \) doesn’t contain the empty set.

Conversely, suppose that \( \mathcal{I} \) does not contain any \( \mathcal{M} \)-unit-valued function. Then \( \emptyset \notin \mathcal{M}(\mathcal{I}) \). By Lemmata 3.4 and 3.5, \( \mathcal{M}(\mathcal{I}) \) is closed under finite intersections. \( \mathcal{M}(\mathcal{I}) \), therefore, satisfies the finite intersection property. By Remark 2.2, \( \mathcal{M}(\mathcal{I}) \) is contained in a filter \( \mathcal{F} \) on \( E \). For this filter, \( \mathcal{I} \subseteq \mathcal{M}_F \), by Remark 3.2.

Ad (2). Suppose \( \mathcal{Q} \) is maximal with respect to not containing any \( \mathcal{M} \)-unit-valued function. By (1), \( \mathcal{Q} \subseteq \mathcal{M}_F \) for some filter \( \mathcal{F} \). Refine \( \mathcal{F} \) to an ultrafilter \( U \).

Then, by Remark 2.5, \( \mathcal{Q} \subseteq \mathcal{M}_F \subseteq \mathcal{M}_U \), and \( \mathcal{M}_U \) doesn’t contain any \( \mathcal{M} \)-unit-valued function. Since \( \mathcal{Q} \) is maximal with this property, \( \mathcal{Q} = \mathcal{M}_U \).

(3) is a special case of (2). \( \square \)

4. A Dichotomy of Maximal Ideals

In what follows, \( D \) is always a domain with quotient field \( K \), \( E \subseteq D \) and \( \mathcal{R} = \mathcal{R}(E, D) \) a ring of functions from \( E \) to \( D \) as in Def. 1.1. When the interpretation of \( \mathcal{R} \) as a subring of \( \prod_{e \in E} D \) is understood, then for \( \mathcal{M} \subseteq D \) we let \( \mathcal{R}(E, \mathcal{M}) = \{ f \in \mathcal{R} \mid f(E) \subseteq \mathcal{M} \} \).

**Proposition 4.1.** Let \( \mathcal{M} \) be a maximal ideal of \( D \) and \( \mathcal{Q} \) a maximal ideal of \( \mathcal{R} = \mathcal{R}(E, D) \). Then exactly one of the following two statements holds:

(1) \( \mathcal{Q} \) contains \( \mathcal{R}(E, \mathcal{M}) = \{ f \in \mathcal{R} \mid f(E) \subseteq \mathcal{M} \} \)

(2) \( \mathcal{Q} \) contains an element \( f \) with \( f(e) \equiv 1 \mod \mathcal{M} \) for all \( e \in E \).

**Proof.** The two cases are mutually exclusive, because any ideal \( \mathcal{Q} \) satisfying both statements must contain 1.

Now suppose \( \mathcal{Q} \) does not contain \( \mathcal{R}(E, \mathcal{M}) \). Let \( g \in \mathcal{R}(E, M) \setminus \mathcal{Q} \). By the maximality of \( \mathcal{Q} \), \( 1 = h(x)g(x) + f(x) \) for some \( h \in \mathcal{R} \) and \( f \in \mathcal{Q} \). We see that \( f(x) = 1 - h(x)g(x) \in \mathcal{Q} \) satisfies \( f(e) \equiv 1 \mod \mathcal{M} \) for all \( e \in E \). \( \square \)

Recall that a function \( f \in \mathcal{R} \) is called \( \mathcal{M} \)-unit-valued if \( f(e) + \mathcal{M} \) is a unit in \( D/\mathcal{M} \) for every \( e \in E \).

**Lemma 4.2.** Let \( \mathcal{M} \) be an ideal of \( D \) and \( \mathcal{Q} \) an ideal of \( \mathcal{R} = \mathcal{R}(E, D) \). The following are equivalent:

(A) \( \mathcal{Q} \) contains an element \( f \) with \( f(e) \equiv 1 \mod \mathcal{M} \) for all \( e \in E \).

(B) \( \mathcal{Q} \) contains an \( \mathcal{M} \)-unit-valued function that takes values in only finitely many residue classes mod \( \mathcal{M} \).

**Proof.** To see that the a priori weaker statement implies the stronger, let \( g \in \mathcal{Q} \) be an \( \mathcal{M} \)-unit-valued function taking only finitely many different values mod \( \mathcal{M} \). Let \( d_1, \ldots, d_k \in D \) be representatives of the finitely many residue classes mod \( \mathcal{M} \) intersecting \( g(E) \) non-trivially, and \( u \in D \) an inverse mod \( \mathcal{M} \) of \( (-1)^{k+1}d_1 \cdots d_k \).
Then
\[ h(x) = \prod_{i=1}^{k} (g(x) - d_i) - (-1)^k d_1 \cdots d_k \]
is in \( \mathbb{Q} \) and \( h(e) \equiv (-1)^{k+1} d_1 \cdots d_k \mod M \) for all \( e \in E \). Therefore \( f(x) = uh(x) \in \mathbb{Q} \) satisfies \( f(e) \equiv 1 \mod M \) for all \( e \in E \).

**Proposition 4.3.** Let \( M \) be a maximal ideal of \( D \) and \( Q \) a maximal ideal of \( \mathcal{R} = \mathcal{R}(E, D) \). If each \( f \in \mathcal{R} \) takes values in only finitely many residue classes \( \mod M \) (in particular, if \( D/M \) happens to be finite) then exactly one of the following statements holds:

1. \( Q \) contains \( \mathcal{R}(E, M) = \{ f \in \mathcal{R} \mid f(E) \subseteq M \} \)
2. \( Q \) contains an \( M \)-unit-valued function.

**Proof.** This follows directly from Proposition 4.1 and Lemma 4.2. □

The Propositions in this section partition the maximal ideals of \( \mathcal{R} \) lying over a maximal ideal \( M \) of \( D \) into two types: those containing \( \mathcal{R}(E, M) \) (the kernel of the restriction to \( \mathcal{R} \) of the canonical projection \( \pi: \prod_{e \in E} D \rightarrow \prod_{e \in E} (D/M) \)), and the others.

In some cases, it is known that all maximal ideals of \( \mathcal{R} \) lying over \( M \) contain \( \mathcal{R}(E, M) \), notably if \( \mathcal{R} = \text{Int}(D) \) and \( M \) is finitely generated and of finite index in \( D \) [1, Ch. V], [4, Lemma 4.4]. We will find a sufficient condition for all maximal ideals of \( \mathcal{R} \) lying over \( M \) to contain \( \mathcal{R}(E, M) \) in Theorem 6.4.

We must not discount the possibility of a maximal ideal \( Q \) lying over \( M \) containing an \( M \)-unit-valued function, however. If \( D \) is an infinite domain, \( D[x] \) is embedded in \( D^D \) by mapping every polynomial to the corresponding polynomial function. When \( D/M \) is not algebraically closed, then there are certainly maximal ideals of \( D[x] \) lying over \( M \) that contain polynomials without a zero \( \mod M \).

### 5. Prime ideals containing \( \mathcal{R}(E, M) \)

We are now in a position to characterize the prime ideals of \( \mathcal{R} \) containing \( \mathcal{R}(E, D) \) as being precisely the ideals of the form \( M_U \) for ultrafilters \( U \) on \( E \), under the following hypothesis: every \( f \in \mathcal{R} \) takes values in only finitely many residue classes of \( M \).

This hypothesis may seem only marginally weaker than the assumption that \( D/M \) is finite. Note however, that it is sometimes satisfied for infinite \( D/M \) under perfectly natural circumstances, for instance, when \( E \) intersects only finitely many residue classes of \( M^n \) for each \( n \in \mathbb{N} \) (\( E \) precompact), and \( \mathcal{R} \) consists of functions that are uniformly \( M \)-adically continuous.

As in the case of integer-valued polynomials, we can show that every prime ideal of \( \mathcal{R}(E, D) \) containing \( \mathcal{R}(E, M) \) is maximal under certain conditions, notably if \( D/M \) is finite. The proof for \( \text{Int}(D) \), when \( D/M \) is finite [1, Lemma V.1.9.], carries over practically without change. Note that Definition 1.1 ensures that every ring
of functions $\mathcal{R}$ contains all constant functions – an essential requirement of the following proof.

**Lemma 5.1.** Let $M$ be a maximal ideal of $D$ such that every function in $\mathcal{R} = \mathcal{R}(E,D)$ takes values in only finitely many residue classes mod $M$, and $Q$ a prime ideal of $\mathcal{R}(E,D)$ containing $\mathcal{R}(E,M)$. Then $Q$ is maximal and $\mathcal{R}/Q$ is isomorphic to $D/M$.

**Proof.** Let $Q$ be a prime ideal of $\mathcal{R}(E,D)$ containing $\mathcal{R}(E,M)$, and $A$ a system of representatives of $D$ mod $M$. It suffices to show that $A$ (viewed as a set of constant functions) is also a system of representatives of $\mathcal{R}$ mod $Q$. Let $f \in \mathcal{R}(E,D)$ and $a_1, \ldots, a_r \in A$ the representatives of those residue classes of $M$ that intersect $f(E)$ non-trivially. Then $\prod_{i=1}^r (f - a_i)$ is in $\mathcal{R}(E,M) \subseteq Q$ and, $Q$ being prime, one of the factors $(f - a_i)$ must be in $Q$. This shows that $f$ is congruent mod $Q$ to one of the constant functions $a_1, \ldots, a_r$, and, in particular, to an element of $A$. Therefore, $A$ is a system of representatives of $\mathcal{R}(E,D)$ mod $Q$. □

**Lemma 5.2.** Let $\mathcal{R} = \mathcal{R}(E,D)$ a ring of functions and $M$ a maximal ideal of $D$ such that every $f \in \mathcal{R}$ takes values in only finitely many residue classes of $M$. Let $\mathcal{I}$ be an ideal of $\mathcal{R}$.

Then $\mathcal{I}$ is contained in an ideal of the form $M_F$ for a filter $F$ on $E$ if and only if $\mathcal{R}(E,M) \subseteq \mathcal{I}$.

**Proof.** $\mathcal{R}(E,M) \subseteq \mathcal{I}$ is equivalent to $\mathcal{I}$ not containing an $M$-unit-valued function, by Proposition 4.3. The statement therefore follows from part (1) of Theorem 3.7. □

**Theorem 5.3.** Let $\mathcal{R} = \mathcal{R}(E,D)$ a ring of functions, and $M$ a maximal ideal of $D$. If every $f \in \mathcal{R}$ takes values in only finitely many residue classes of $M$ (and, in particular, if $D/M$ is finite), then the prime ideals of $\mathcal{R}$ containing $\mathcal{R}(E,M)$ are exactly the ideals of the form $M_U$ with $U$ an ultrafilter on $E$. Each of them is maximal and its residue field isomorphic to $D/M$.

**Proof.** Let $Q$ be a prime ideal of $\mathcal{R}$ containing $\mathcal{R}(E,M)$. By Lemma 5.1, $Q$ is maximal and $\mathcal{R}/Q$ is isomorphic to $D/M$. By Lemma 5.2, $Q \subseteq M_F$ for some filter $F$ on $E$. $F$ can be refined to an ultrafilter $U$ on $E$, and then $Q \subseteq M_F \subseteq M_U \neq \mathcal{R}$, by Remark 2.5. Since $Q$ is maximal, $Q = M_U$ follows.

Conversely, every ideal of the form $M_U$ for an ultrafilter $U$ on $E$ is prime, by Lemma 2.6, and contains $\mathcal{R}(E,M)$, by Remark 2.5. □

Note, in particular, that Theorems 3.7 and 5.3 apply to $\mathcal{R} = \text{Int}(E,D)$. In this way, we see, when $M$ is a maximal ideal of finite index in $D$, that prime ideals of $\text{Int}(E,D)$ containing $\text{Int}(D,M)$ are inverse images of prime ideals of $D^E$, and ultimately come from ultrapowers of $(D/M)$, as in the discussion after Lemma 2.6.
6. DIVISIBLE RINGS OF FUNCTIONS

Let \( \mathcal{R} \subseteq D^E \) be a ring of functions and \( M \) a maximal ideal of \( D \). We have seen that we can describe those maximal ideals of \( \mathcal{R} \) lying over \( M \) that contain \( \mathcal{R}(E,M) \). We would like to know under what conditions this holds for every maximal ideal of \( \mathcal{R} \) lying over \( M \).

In the case where \( M \) is a maximal ideal of finite index in a one-dimensional Noetherian domain \( D \), Chabert showed that every maximal ideal of \( \text{Int}(D) \) lying over \( M \) contains \( \text{Int}(D,M) \), cf. [1, Prop. V.1.11] and [4, Lemma 3.3]. Once we know this, Theorem 5.3 is applicable. It can be used to give an alternative proof of the fact that every prime ideal of \( \text{Int}(D) \) lying over \( M \) is maximal and of the form \( M_{\alpha} = \{ f \in \text{Int}(D) \mid f(\alpha) \in \hat{M} \} \) for an element \( \alpha \) in the \( M \)-adic completion of \( D \).

We will now generalize Chabert’s argument from integer-valued polynomials to a class of rings of functions which we call divisible. Note that we do not have to restrict ourselves to Noetherian domains; we only require the individual maximal ideal for which we study the primes of \( \mathcal{R} \) lying over it to be finitely generated. It is true that our questions only localize well when the domain is Noetherian, but we will pursue a different course, not relying on localization.

**Definition 6.1.** Let \( R \) be a commutative ring and \( E \subseteq R \). We call a ring of functions \( \mathcal{R} \subseteq R^E \) **divisible** if it has the following property: If \( f \in \mathcal{R} \) is such that \( f(E) \subseteq cR \) for some non-zero \( c \in R \), then every function \( g \in R^E \) satisfying \( cg(x) = f(x) \) is also in \( \mathcal{R} \).

We call \( \mathcal{R} \) **weakly divisible** if for every \( f \in \mathcal{R} \) and every non-zero \( c \in R \) such that \( f(E) \subseteq cR \), there exists a function \( g \in \mathcal{R} \) with \( cg(x) = f(x) \).

If \( R \) is a domain, we note that \( g(x) \) in the above definition is unique and that, therefore, for domains, weakly divisible is equivalent to divisible.

**Example 6.2.**

1. \( \text{Int}(E,D) \) is divisible. - This is our motivation.

2. If \( D \) is a valuation domain with maximal ideal \( M \) then the ring of uniformly \( M \)-adically continuous functions from \( E \) to \( D \) (\( E \subseteq D \) equipped with subspace topology of \( M \)-adic topology) is a divisible ring of functions.

We now consider minimal prime ideals of non-zero principal ideals, that is, \( P \) containing some \( p \neq 0 \) such that there is no prime ideal strictly contained in \( P \) and containing \( p \). If \( D \) is Noetherian, this condition reduces to “\( \text{ht}(P) = 1 \)”. In non-Noetherian domains, we find examples with \( \text{ht}(P) > 1 \), for instance, the maximal ideal of a finite-dimensional valuation domain.

**Lemma 6.3.** Let \( R \) be a domain, \( P \) a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal \( (p) \subseteq P \). Then there exist \( m \in \mathbb{N} \) and \( s \in R \setminus P \) such that \( sp^m \subseteq pR \).

**Proof.** In the localization \( R_P \), \( P_P \) is the radical of \( pR_P \). Therefore, since \( P \) (and hence \( P_P \)) is finitely generated, there exists \( m \in \mathbb{N} \) with \( P_P^m \subseteq pR_P \) and in
particular $P^m \subseteq pR$. The ideal $P^m$ is also finitely generated, by $p_1, \ldots, p_k$, say. Let $a_i \in R$ with $p_i = pa_i$. By considering the fractions $a_i = r_i/s_i$ (with $r_i \in R$ and $s_i \in R \setminus P$), and setting $s = s_1 \cdots s_k$, we see that $sP^m \subseteq pR$ as desired. □

**Theorem 6.4.** Let $D$ be a domain and $P$ a finitely generated prime ideal that is a minimal prime of a non-zero principal ideal. Let $R \subseteq D^E$ be a divisible ring of functions from $E$ to $D$. Then every prime ideal $Q$ of $R$ with $Q \cap D = P$ contains $R(E,P)$.

**Proof.** Let $f \in R(E,P)$. Let $p \in P$ non-zero and such that there is no prime ideal $P_1$ with $(p) \subseteq P_1 \subsetneq P$. By Lemma 6.3, there are $m \in \mathbb{N}$ and $s \in D \setminus P$ such that $s^m p \subseteq pD$. Then $sf^m \in R(E,pD)$. Since $R$ is divisible, $sf^m = pg$ for some $g \in R(E,D)$. Therefore, $sf^m \in pR(E,D) \subseteq Q$. As $Q$ is prime and $s \notin Q$, we conclude that $f \in Q$. □

**Corollary 6.5.** Let $D$ be a domain, $M$ a finitely generated maximal ideal of height 1, and $E$ a subset of $D$. Let $R \subseteq D^E$ be a divisible ring of functions from $E$ to $D$, such that each $f \in R$ takes its values in only finitely many residue classes of $M$ in $D$.

Then the prime ideals of $R$ lying over $M$ are precisely the ideals of the form $M_U$ for an ultrafilter $U$ on $E$. Each $M_U$ is a maximal ideal and its residue field isomorphic to $D/M$.

**Proof.** This follows from Theorem 6.4 via Theorem 5.3. □

To summarize, we can, using ultrafilters, describe certain prime ideals of a ring of functions $R = R(E,D)$ lying over a maximal ideal $M$ pretty well: namely, those prime ideals that do not contain $M$-unit-valued functions (Theorem 3.7), or that contain $R(E,M)$ (Theorem 5.3).

We have, so far, little information about when all prime ideals of $R$ lying over $M$ are of this form, apart from the sufficient condition in Theorem 6.4.

If we restrict our attention to rings of functions $R$ with $D[x] \subseteq R(E,D) \subseteq D^E$, it would be interesting to find a precise criterion, perhaps involving topological density, for this property.

Note that in the “nicest” case, that of $\text{Int}(D)$, where $D$ is a Dedekind ring with finite residue fields, not only is $\text{Int}(D,M)$ contained in every prime ideal of $\text{Int}(D)$ lying over a maximal ideal $M$ of $D$, but also $\text{Int}(D)$ is dense in $D^D$ with product topology of discrete topology on $D$ [2,3].

**References**


Department of Analysis and Number Theory (5010), Technische Universität Graz, Kopernikusgasse 24, 8010 Graz, Austria

E-mail address: frisch@math.tugraz.at