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NULLSTELLENSATZ AND SKOLEM PROPERTIES FOR INTEGER-VALUED POLYNOMIALS

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Abstract. Skolem and Nullstellensatz properties are analogues of the weak Nullstellensatz and Hilbert's Nullstellensatz, respectively, for the ring of integer-valued polynomials in several indeterminates $\mathrm{Int}(D^n) = \{f \in K[x_1, \ldots, x_n] \mid f(D^n) \subseteq D\}$, where D is a domain and K its quotient field. We show their equivalence when D is a Noetherian domain and extend the criterion of Brizolis and Chabert for $\mathrm{Int}(D^n)$ to have the Nullstellensatz property to all Noetherian domains D.

I. *Introduction and Definitions*

Hilbert's Nullstellensatz states for an algebraically closed field F that if $f_1, \ldots, f_m$ and f are in $F[x_1, \ldots, x_n]$ such that $f(\underline{a}) = 0$ for all those $\underline{a} \in F^n$ for which $f_1(\underline{a}) = \ldots = f_m(\underline{a}) = 0$, then $f \in \sqrt{(f_1, \ldots, f_m)}$. In this paper we investigate analogues of this theorem that hold when we replace F by a Noetherian domain D and $F[x_1, \ldots, x_n]$ by the ring of integer-valued polynomials in n indeterminates over D. For a survey of this topic, see chapters VII and XI of [2], and for related recent research [3, 8, 9, 5]. Throughout this paper, D denotes an integral domain with quotient field K.

Definition. The ring of integer-valued polynomials in n indeterminates over D is defined as $\mathrm{Int}(D^n) = \{f \in K[x_1, \ldots, x_n] \mid f(\underline{a}) \in D \textit{ for all } \underline{a} \in D^n\}$. We write $\mathrm{Int}(D)$ for $\mathrm{Int}(D^1)$, and define $\mathrm{Int}(D^0) = D$.

Brizolis [1] in 1975 proposed the following "Nullstellensatz property" and characterized among Dedekind domains with perfect residue fields those D for which $\mathrm{Int}(D^n)$ had the property (including, for instance, the ring of integers of every number field).

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Definition. $\mathrm{Int}(D^n)$ has the *Nullstellensatz property*, if for all polynomials f_1, $\ldots$, f_m and $f \in \mathrm{Int}(D^n)$ satisfying $f(\underline{a}) \in (f_1(\underline{a}), \ldots, f_m(\underline{a}))$ for all $\underline{a} \in D^n$, it is true that $f \in \sqrt{(f_1, \ldots, f_m)}$.

We also consider a slightly stronger property, which we accordingly call the stronger Nullstellensatz property. For an ideal I in a commutative ring R, we denote by $\mathcal{J}(I)$ the Jacobson radical of I, i.e., the intersection of all maximal ideals of R containing I.

Definition. We say that $\mathrm{Int}(D^n)$ has the *stronger Nullstellensatz property*, if for all polynomials $f_1, \ldots, f_m$ and $f \in \mathrm{Int}(D^n)$ satisfying $f(\underline{a}) \in \mathcal{J}\big((f_1(\underline{a}), \ldots, f_m(\underline{a}))\big)$ for all $\underline{a} \in D^n$, it is true that $f \in \sqrt{(f_1, \ldots, f_m)}$.

In 1936, Skolem [16] had shown an analogue for integer-valued polynomials of the so called "weak Nullstellensatz", the theorem stating for an algebraically closed field F that if f_1, $\ldots$, f_m generate a proper ideal of $F[x_1, \ldots, x_n]$ then there exists $\underline{a} \in F^n$ with $f_1(\underline{a}) = \ldots = f_m(\underline{a}) = 0$. Skolem gave a construction of $g_1, \ldots, g_m \in \mathrm{Int}(\mathbb{Z}^n)$ with $f_1 g_1 + \ldots + f_m g_m = 1$ for any $f_1, \ldots, f_m \in \mathrm{Int}(\mathbb{Z}^n)$ satisfying $f_1(\underline{a})\,\mathbb{Z} + \ldots + f_m(\underline{a})\,\mathbb{Z} = \mathbb{Z}$ for all $\underline{a} \in \mathbb{Z}^n$, thereby effectively proving for $\mathrm{Int}(\mathbb{Z}^n)$ the condition since known as the Skolem property:

Definition. $\mathrm{Int}(D^n)$ has the *Skolem property*, if for every finite set of polynomials $f_1, \ldots, f_m \in \mathrm{Int}(D^n)$ satisfying $(f_1(\underline{a}), \ldots, f_m(\underline{a})) = D$ for all $\underline{a} \in D^n$, it is true that $(f_1, \ldots, f_m) = \mathrm{Int}(D^n)$. D is called a *Skolem ring* if the ring of integer-valued polynomials in one indeterminate, $\mathrm{Int}(D)$, has the Skolem property.

Clearly, the Nullstellensatz property for $\mathrm{Int}(D^n)$ implies the Skolem property for $\mathrm{Int}(D^n)$. Also, it is easy to see that each of the properties for $\mathrm{Int}(D^n)$ implies the same property for $\mathrm{Int}(D^m)$ for all $m \le n$ ([2] Lemma XI.3.2). In particular, if $\mathrm{Int}(D^n)$ satisfies the Skolem property for some n then D is a Skolem ring.

Remark 1.1. We must bear in mind that the Skolem property of $\mathrm{Int}(D^n)$ depends not only on the ring $\mathrm{Int}(D^n)$, but also on the manner of its elements being interpreted as functions. As a ring, $\mathrm{Int}(D^{n+1})$ is canonically isomorphic to $\mathrm{Int}(\mathrm{Int}(D^n))$ for every infinite domain D ([2] Prop. XI.1.1), but the Skolem property of $\mathrm{Int}(D^{n+1})$ (concerning functions in $n+1$ variables $f\colon D^{n+1} \to D$) is different from the Skolem property of $\mathrm{Int}(\mathrm{Int}(D^n))$ (concerning functions in one variable $f\colon \mathrm{Int}(D^n) \to \mathrm{Int}(D^n)$). It is, however, easy to see that the former implies the latter (cf. Remark 4.3).

Of the various necessary and sufficient conditions for Skolem and Nullstellensatz properties proved by Brizolis, Chabert and McQuillan, the most general so far is Chabert's result [7] that a certain list of properties, which we will call the "Chabert-Brizolis criterion" for short (condition 5 of Theorem 4.2 below), characterizes the domains D for which $\mathrm{Int}(D^n)$ has the Nullstellensatz property for all n, both among Noetherian domains with $\dim(D) > 1$ and among Noetherian domains of $\dim(D) = 1$ and $\mathrm{char}(D) = 0$. This characterization implies the equivalence of Skolem and (a priori stronger) Nullstellensatz properties for those Noetherian domains for which it holds, since the criterion is known to be necessary for D to be a Skolem ring.

In fact, it is known that a Noetherian domain satisfies the Chabert-Brizolis criterion if and only if it is a Skolem ring. This previous result of Chabert [6] and McQuillan [15], together with the equivalence of Skolem and Nullstellensatz properties in one variable (Chabert [7] and Brizolis [1]), shows that the characterization we wish to prove for several variables is true for one variable. Due to the fact that $K[x]$ is a principal ideal domain, the one variable case can be proved with comparatively little effort (cf. [2], Theorems VII.5.1 and VII.5.9).

In this paper we show for an arbitrary Noetherian domain D the equivalence of the Chabert-Brizolis criterion to the Nullstellensatz property of $\mathrm{Int}(D^n)$ for all n, thereby closing the gap at $\dim(D) = 1$, $\mathrm{char}(D) \neq 0$. Our proof takes its simplest form in the case of a one-dimensional domain, but works for any Noetherian domain with the addition of a few technicalities (Remark 2.3, Lemma 3.3, Lemma 3.4) that can be skipped when assuming $\dim(D) = 1$.

We use a two-step approach showing first the equivalence of Skolem and Nullstellensatz property (Theorem 2.4) and then the sufficiency of the criterion for the Skolem property (Theorem 3.5). For the first step, we adapt Rabinowitsch's argument that the weak Nullstellensatz for $F[x_1, \ldots, x_{n+1}]$ implies the Nullstellensatz for $F[x_1, \ldots, x_n]$, to show that the Skolem property for $\mathrm{Int}(D^{n+1})$ implies the Nullstellensatz property for $\mathrm{Int}(D^n)$.

To formulate the Chabert-Brizolis criterion, we must define two ring-theoretic entities, the well-known Hilbert ring (a.k.a. Jacobson ring), and the lesser known d-ring. The criterion then consists of these two properties, together with the requirement that every maximal ideal M of D either have an algebraically closed residue field or be of height 1 with a finite residue field.

Definition. A domain D is a *Hilbert ring* (Goldman [11]), if the radical of every ideal I of D is equal to $\mathcal{J}(I)$, the intersection of all maximal ideals containing I. (Krull called such rings *Jacobson rings* [14].)

Although Hilbert rings play an important part in the investigation of Nullstellensatz analogues for integer-valued polynomials, so far no analogue has been proposed for Goldman's and Krull's proof of the Nullstellensatz ([13] §1.3) based on the fact that a domain R is a Hilbert ring if and only if $R[x]$ is a Hilbert ring. While in principle it is possible to get to $\mathrm{Int}(D^n)$ from D inductively by repeatedly passing from R to $\mathrm{Int}(R)$ (by the isomorphism $\mathrm{Int}(\mathrm{Int}(D^n)) \simeq \mathrm{Int}(D^{n+1})$), no condition comparable to the Hilbert ring property is known to pass from R to $\mathrm{Int}(R)$. In particular, R being a Hilbert ring does not imply the same for $\mathrm{Int}(R)$ ([1] Example 4.3).

Brizolis [1] and Gunji and McQuillan [12] simultaneously introduced d-rings in 1975. This is the concept needed to show the Skolem property with respect to ideals of $\mathrm{Int}(D^n)$ lying over (0) in D.

Definition. A domain D is a *d-ring* if for every non-constant polynomial $f \in D[x]$ there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.

Alternatively, d-rings can be characterized as those domains, for which every integer-valued rational function is an integer-valued polynomial. We summarize a few facts about d-rings that we'll need (cf. [2], §VII.2).

Fact 1.2 ([1] Lemma 1.3, [12] Prop. 1). The following are equivalent:

(i) D is a d-ring.

(ii) For every non-constant $f \in D[x]$, if $\mathcal{S}$ is the set of maximal ideals M of D, such that f has a zero mod M, then $\bigcap_{M\in\mathcal{S}} M = (0)$.

(iii) For every non-constant $f \in \mathrm{Int}(D)$, there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.

Fact 1.3 ([12] Prop. 3, Corollary 2). Let $D \subseteq R$ be domains, and D a d-ring.

(a) If R is integral over D, then R is a d-ring.

(b) If R is finitely generated as a ring over D, then R is a d-ring.

Note that Fact 1.2 (ii) implies that (0) in a d-ring is always an intersection of maximal ideals, and that therefore every one-dimensional d-ring is a Hilbert ring.

II. *Equivalence of Nullstellensatz and Skolem properties*

Let M be a maximal ideal of D. For every $\underline{a} \in D^n$, there is a maximal ideal of $\mathrm{Int}(D^n)$ lying over M (with the same residue field as M) given by

$$\mathcal{M}_{\underline{a},M} = \{f \in \mathrm{Int}(D^n) \mid f(\underline{a}) \in M\}.$$

$\mathrm{Int}(D^n)$ has the Skolem property if and only if every finitely generated proper ideal of $\mathrm{Int}(D^n)$ is contained in a maximal ideal of the form $\mathcal{M}_{\underline{a},M}$, for some maximal ideal M of D and $\underline{a} \in D^n$. Similarly, $\mathrm{Int}(D^n)$ has the stronger Nullstellensatz property if and only if the radical of every finitely generated ideal I of $\mathrm{Int}(D^n)$ is an intersection of maximal ideals of the form $\mathcal{M}_{\underline{a},M}$, with M a maximal ideal of D and $\underline{a} \in D^n$.

Since prime ideals play a major rôle in the following, we briefly describe $\mathrm{Spec}(\mathrm{Int}(D^n))$. First recall that if P is a prime ideal of infinite index in D, then $\mathrm{Int}(D^n) \subseteq D_P[x_1, \ldots, x_n]$ ([2], Corollary XI.1.11) and therefore

$$(D \setminus P)^{-1}\mathrm{Int}(D^n) = D_P[x_1, \ldots, x_n],$$

which implies a bijective correspondence (given by lying over) between all prime ideals of $D_P[x_1, \ldots, x_n]$ and those prime ideals of $\mathrm{Int}(D^n)$ whose intersection with D is contained in P.

Fact 2.1. *Let D be a domain and P a prime ideal of infinite index in D. The prime ideals Q of $\mathrm{Int}(D^n)$ with $Q \cap D = P$ are in bijective correspondence, via*

$$Q = \mathcal{Q} \cap \mathrm{Int}(D^n), \qquad (\mathcal{Q} \text{ prime} \trianglelefteq D_P[x_1, \ldots, x_n],\ PD_P[x_1, \ldots, x_n] \subseteq \mathcal{Q}),$$

to the prime ideals $\mathcal{Q}$ of $D_P[x_1, \ldots, x_n]$ containing $PD_P[x_1, \ldots, x_n]$, and therefore in bijective correspondence to the prime ideals $\tilde{\mathcal{Q}}$ of $(D_P/PD_P)[x_1, \ldots, x_n]$, via

$$Q = \pi^{-1}(\tilde{\mathcal{Q}}) \cap \mathrm{Int}(D^n), \quad (\tilde{\mathcal{Q}} \text{ prime } \trianglelefteq (D_P/PD_P)[x_1, \ldots, x_n]),$$

where $\pi\colon D_P[x_1, \ldots, x_n] \to (D_P/PD_P)[x_1, \ldots, x_n]$ is the canonical projection.

The prime ideals of $\mathrm{Int}(D^n)$ above a maximal ideal M of finite index in D are not as easily characterized, but if $\mathrm{ht}(M) = 1$, they are known. First note that for every ideal M of D, every polynomial in $\mathrm{Int}(D^n)$ is uniformly continuous in M-adic topology. For Noetherian D, this is shown in [2], Prop. III.2.3, by means of the Artin-Rees Lemma; an elementary proof for arbitrary domains can be found in [4] Prop. 1.4 (cf. also Remark 1.5). Both proofs are for one variable, but obviously carry over to several. Therefore, if M is a maximal ideal of D, and $\widehat{D_M}$ the M-adic completion of D, then every $f \in \mathrm{Int}(D^n)$ induces a continuous function $f\colon \widehat{D_M}^n \to \widehat{D_M}$.

Fact 2.2 ([2], Corollary XI.1.8 and Lemma XI.2.8). *Let D be a Noetherian domain and M a maximal ideal of finite index in D with $\mathrm{ht}(M)=1$. By $\widehat{D_M}$ denote the M-adic completion of D with maximal ideal $\hat{M}$. Then every prime ideal of $\mathrm{Int}(D^n)$ above M is maximal and of the form*

$$\mathcal{M}_{\underline{a},\hat{M}} = \{f \in \mathrm{Int}(D^n) \mid f(\underline{a}) \in \hat{M}\},$$

for some $\underline{a} \in \widehat{D_M}^{\,n}$.

Remark 2.3 ([2] Lemma VII.4.2). If M is a maximal ideal in a Noetherian Skolem ring, then D/M is algebraically closed or $\mathrm{ht}(M)=1$.

Proof. Suppose otherwise. (This implies D is not a field.) Since $\mathrm{ht}(M)>1$, there exists a maximal ideal M' of the integral closure D' of D with $M'\cap D=M$ and $\mathrm{ht}(M')>1$. D being Noetherian, D' is a Krull domain and so is $D'_{M'}$, such that $D'_{M'}=\bigcap_{P'\subseteq M'} D'_{P'}$, the intersection being over all height 1 prime ideals of D' contained in M'. Each such P' is properly contained in M', so $P=P'\cap D$ is properly contained in M and therefore of infinite index in D, such that $\mathrm{Int}(D)\subseteq D_P[x]\subseteq D'_{P'}[x]$. We have shown $\mathrm{Int}(D)\subseteq D'_{M'}[x]$.

As D/M is not algebraically closed, there exists a polynomial $f\in D[x]$, not a constant in $(D/M)[x]$, that has no zero mod M, wherefore by the Skolem property, $(M,f)\mathrm{Int}(D)=\mathrm{Int}(D)$. This implies $(M,f)D'_{M'}[x]=D'_{M'}[x]$, making f a constant in $(D'_{M'}/M'D'_{M'})[x]$, contrary to assumption. $\square$

Theorem 2.4. *Let D be a Noetherian domain and $n\in\mathbb{N}$. If $\mathrm{Int}(D^{n+1})$ has the Skolem property, then $\mathrm{Int}(D^n)$ has the stronger Nullstellensatz property.*

Proof. Given $f_1,\ldots,f_m,f\in\mathrm{Int}(D^n)$ such that for every maximal ideal M of D, and every $\underline{a}\in D^n$, whenever $f_1(\underline{a}),\ldots,f_m(\underline{a})\in M$ then $f(\underline{a})\in M$, we must show for every prime ideal Q of $\mathrm{Int}(D^n)$: if $f_1,\ldots,f_m\in Q$ then $f\in Q$.

First consider a prime ideal Q lying over a prime ideal of infinite index in D. By Fact 2.1, $Q=\mathrm{Int}(D^n)\cap\mathcal{Q}$, where $\mathcal{Q}$ is a prime ideal of $R[x_1,\ldots,x_n]$ for an overring R of D, such that $\mathrm{Int}(D^n)\subseteq R[x_1,\ldots,x_n]$ and $\mathrm{Int}(D^{n+1})\subseteq R[x_1,\ldots,x_{n+1}]$.

The polynomials $f_1,\ldots,f_m,1-x_{n+1}f\in\mathrm{Int}(D^{n+1})$ satisfy the hypothesis of the Skolem property of $\mathrm{Int}(D^{n+1})$, so there exist $h_1,\ldots,h_m,h\in\mathrm{Int}(D^{n+1})$ with

$$f_1h_1+\ldots+f_mh_m+(1-x_{n+1}f)h=1.$$

We now substitute $1/f$ for x_{n+1} and multiply both sides by f^r, where r is the maximal degree in x_{n+1} of any h_i. This gives $f_1g_1+\ldots+f_mg_m=f^r$, where $g_1,\ldots,g_m\in R[x_1,\ldots,x_n]$, because h_i is of the form $\sum c_{k_1,\ldots,k_{n+1}}x^{k_1}\ldots x^{k_{n+1}}$, with

$c_{k_1,\dots,k_{n+1}} \in R$ and therefore $g_i = \sum c_{k_1,\dots,k_{n+1}} x^{k_1} \dots x^{k_n} f^{r-k_{n+1}}$, with $c_{k_1,\dots,k_{n+1}} \in R$ and $f^{r-k_{n+1}} \in R[x_1,\dots,x_n]$. Therefore $f_1,\dots,f_m \in \mathcal{Q}$ implies $f^r \in \mathcal{Q}$, and, $\mathcal{Q}$ being prime, $f \in \mathcal{Q}$. We conclude that $f \in Q = \mathrm{Int}(D^n) \cap \mathcal{Q}$.

Now consider a prime ideal Q of $\mathrm{Int}(D^n)$ above a maximal ideal M of finite index in D. By Fact 2.2 and Remark 2.3, $Q = \mathcal{M}_{\underline{a},\hat{M}} = \{g \in \mathrm{Int}(D^n) \mid g(\underline{a}) \in \hat{M}\}$, for some $\underline{a} \in \widehat{D_M}^n$. If $f_1,\dots,f_m \in \mathcal{M}_{\underline{a},\hat{M}}$, then, since the functions $f_i \colon \widehat{D_M}^n \to \widehat{D_M}$ are continuous, some $\hat{M}$-adic neighborhood U of $\underline{a}$ gets mapped into $\hat{M}$ by all f_i. In particular, $f_1(\underline{b}),\dots,f_m(\underline{b}) \in M$ for all $\underline{b} \in U \cap D^n$, and therefore, by the hypothesis of the stronger Nullstellensatz property, $f(\underline{b}) \in M$ for all $\underline{b} \in U \cap D^n$. Since D is dense in $\widehat{D_M}$, the continuity of $f \colon \widehat{D_M}^n \to \widehat{D_M}$ implies $f(\underline{a}) \in \hat{M}$, i.e. $f \in \mathcal{M}_{\underline{a},\hat{M}}$. $\square$

Corollary. *If D is a Noetherian domain then $\mathrm{Int}(D^n)$ has the stronger Nullstellensatz property for all $n \in \mathbb{N}$ if and only if $\mathrm{Int}(D^n)$ has the Skolem property for all $n \in \mathbb{N}$.*

The Rabinowitsch-type argument from the first case of the above proof also yields a simpler proof of the following fact, which forms part of the argument for the necessity of the Chabert-Brizolis criterion:

Remark 2.5 ([2], Thm. VII.5.1). Every Noetherian Skolem ring is a Hilbert ring; more generally, the radical of every finitely generated ideal in a Skolem ring is an intersection of maximal ideals.

Proof. Given $d_1,\dots,d_m$ and $d \in D$, such that $d_1,\dots,d_m \in M \Longrightarrow d \in M$, for every maximal ideal M of D, and a non-maximal prime ideal P of D with $d_1,\dots,d_m \in P$, we use the Skolem property of $\mathrm{Int}(D)$ to show $d \in P$:

Since $[D:P]$ is infinite, $\mathrm{Int}(D) \subseteq D_P[x]$. The polynomials $d_1,\dots,d_m$ and $1-dx$ in $\mathrm{Int}(D)$ satisfy the hypothesis of the Skolem property, so there exist $g_1,\dots,g_m$ and $g \in \mathrm{Int}(D)$ with $d_1g_1+\dots+d_mg_m+(1-dx)g=1$. We substitute d^{-1} for x and multiply both sides by d^r, where $r = \max\deg g_i$, and get $d_1h_1+\dots+d_mh_m = d^r$, with $h_i \in K$. Since $g_i(x) = \sum_k a_k x^k$ with $a_k \in D_P$, however, $h_i = \sum_k a_k d^{r-k}$ is actually in D_P. Therefore, $d_1,\dots,d_m \in P$ implies $d^r \in PD_P$, and we conclude $d \in PD_P \cap D = P$. $\square$

III. *Sufficiency of Criterion for Skolem Property*

We proceed to show for Noetherian domains of arbitrary characteristic that the Chabert-Brizolis criterion implies the Skolem property in several indeterminates (Theorem 3.5). The nontrivial part of the argument concerns maximal ideals of

$\mathrm{Int}(D^n)$ lying over (0) in D (Lemma 3.2). To work around the problems presented by non-separable extensions of K we will replace K by its prefect closure F, whose algebraic extensions are all separable. This necessitates replacing D by a finite ring extension inside F, but, owing to the restrictions on the residue fields of D, this ring extension turns out to be harmless for our purposes, since it involves only few nontrivial residue field extensions. (Lemma 3.1).

Recall that for every field K there exists in the algebraic closure $\bar{K}$ a unique smallest perfect field containing K, called the *perfect closure* of K. If $\mathrm{char}(K)=0$, then K is its own prefect closure; if $\mathrm{char}(K) = p$, then the perfect closure is obtained from K by adjoining the p^n-th roots (for all $n \in \mathbb{N}$) of all elements of K. This is a consequence of the fact that a field E of characteristic p is perfect if and only if the Frobenius endomorphism $\varphi(x) = x^p$ on E, $\varphi: E \to E$, is surjective.

The following two lemmata are mainly relevant in characteristic p, as the contingency they provide for, a maximal ideal $\mathcal{M}$ of $\mathrm{Int}(D^n)$ with $\mathcal{M} \cap D = (0)$, is known not to occur for a Noetherian d-ring of characteristic 0 ([2], Lemma XI.3.3, Prop. XI.3.4). Allowing $\mathrm{char}(D) = 0$, however, does not add any difficulty and keeps our proof self-contained.

Lemma 3.1. *Let D be a domain with quotient field K, F the perfect closure of K, and $R = D[c_1, \dots, c_k] \subseteq F$ a domain finitely generated as a ring over D. Let $\mathcal{S}$ denote the set of prime ideals P of D, such that D_P/PD_P is a perfect field and for some prime ideal Q of R with $Q \cap D = P$ the embedding of residue fields $D_P/PD_P \hookrightarrow R_Q/QR_Q$ is not surjective. Then $\bigcap_{P\in\mathcal{S}} P \neq (0)$.*

Proof. First assume $\mathrm{char}(K) = p$. Since $R \subseteq F$, there exists an $n \in \mathbb{N}$ such that $c_i^{p^n} \in K$ for $1 \le i \le k$. Write $c_i^{p^n} = a_i b_i^{-1}$ with a_i, $b_i \in D$ and set $b = b_1 \cdot \ldots \cdot b_k$. We claim that $b \in \bigcap_{P\in\mathcal{S}} P$. Suppose P is a prime ideal of D such that $b \notin P$ and D_P/PD_P is perfect. Let Q be a prime ideal of R with $Q \cap D = P$; we must show that $D_P/PD_P \hookrightarrow R_Q/QR_Q$ is surjective.

The residue classes of $c_1, \dots, c_k$ generate R_Q/QR_Q over D_P/PD_P, and c_i is a p^n-th root of $a_i b_i^{-1} \in D_P$. Now $a_i b_i^{-1} + PD_P$ has a p^n-th root $d_i + PD_P$ in D_P/PD_P, since this field is perfect. By the injectivity of the Frobenius homomorphism, $c_i + QR_Q = d_i + QR_Q$, i.e., $c_i + QR_Q$ is in the image of $D_P/PD_P \hookrightarrow R_Q/QR_Q$.

If $\mathrm{char}(K) = 0$, then $R \subseteq K$ and $c_i = a_i b_i^{-1}$ with $a_i, b_i \in D$. Let $b = b_1 \cdot \ldots \cdot b_k$. Then $c_1, \dots, c_k \in D_P$ for every prime ideal P of D with $b \notin P$, i.e., any residue field extension $D_P/PD_P \hookrightarrow R_Q/QR_Q$ with $b \notin P$ is trivial. $\square$

Lemma 3.2. *Let D be a d-ring, such that the residue field of every maximal ideal of D is perfect. If I is a finitely generated ideal of $\mathrm{Int}(D^n)$ contained in an ideal of the form $\mathcal{M}=\mathcal{P}\cap\mathrm{Int}(D^n)$, where $\mathcal{P}=\{f\in K[x_1,\ldots,x_n]\mid f(u_1,\ldots,u_n)=0\}$, $u_1,\ldots,u_n\in\bar{K}$, then $I\subseteq\mathcal{M}_{\underline{a},M}$ for some maximal ideal M of D and $\underline{a}\in D^n$.*

Proof. Let F be the perfect closure of K, and $u\in\bar{K}$ such that $F[u_1,\ldots,u_n]=F[u]$. By multiplying u with a suitable constant in D, if necessary, we may assume u integral over D. For $1\le i\le n$, let $g_i(x)\in F[x]$, such that $g_i(u)=u_i$. If $c_1,\ldots,c_l\in F$ is a list of the coefficients of $g_1,\ldots,g_n$, let $d\in D$ such that $dc_i=c_i'$ is integral over D for $1\le i\le l$ and set $E=K[c_1,\ldots,c_l]=K[c_1',\ldots,c_l']$.

Let $f(x)=x^k+\alpha_{k-1}x^{k-1}+\ldots+\alpha_0$ be the minimal polynomial of u over E. Its coefficients $\alpha_0,\ldots,\alpha_{k-1}$ are integral over D, since u is. Define $R=D[c_1',\ldots,c_l',\alpha_0,\ldots,\alpha_{k-1}]$, then R is a domain with quotient field E, finitely generated integral over D, and a d-ring by Fact 1.3.

Let $f_1,\ldots,f_m$ be generators of I, and define $f_i^*=f_i(g_1(x),\ldots,g_n(x))\in E[x]$. Since $f_i^*(u)=f_i(u_1,\ldots,u_n)=0$, there exist $h_i\in E[x]$ with $f_i^*(x)=f(x)h_i(x)$. Let $\beta\in R$ be a common denominator of the coefficients (written as fractions of elements in R) of all the polynomials g_j, f_i, f_i^* and h_i for $1\le j\le n$, $1\le i\le m$.

Let $b\in D$ be a non-zero element (existing by Lemma 3.1) in the intersection of all maximal ideals M of D such that $D_M/MD_M\hookrightarrow R_P/PR_P$ is not surjective for some prime ideal P of R with $P\cap D=M$.

Using Fact 1.2 (ii), let Q be a maximal ideal of R with $b\beta\notin Q$, and $r\in R$ with $f(r)\in Q$. Let $M=Q\cap D$; then M is a maximal ideal of D, since R is integral over D and Q is maximal. We claim that $f_1,\ldots,f_m$ have a simultaneous zero mod M in D^n.

Since $\beta\notin Q$, all coefficients of all the polynomials g_j, f_i, f_i^*, h_i are in R_Q (and those of f are in R, by construction). Let $s_j=g_j(r)$ $(1\le j\le n)$, then

$$f_i(s_1,\ldots,s_n)=f_i^*(r)=f(r)h_i(r)\in QR_Q.$$

Let $t_j\in D_M$ with $t_j+QR_Q=s_j+QR_Q$ (exists since $b\notin M$) and let $a_j\in D$ with $a_j+MD_M=t_j+MD_M$ (exists since M is maximal). Then for $1\le i\le m$, $f_i(a_1,\ldots,a_n)\in QR_Q\cap D=M$, or in other words, $f_i\in\mathcal{M}_{\underline{a},M}$. □

The following two lemmata are needed to accommodate domains of $\dim(D)>1$. We show that for those domains, under certain hypotheses (which are always satisfied by a Skolem ring), every maximal ideal of $\mathrm{Int}(D^n)$ lies over a maximal ideal of D. Actually, more is true: under the same hypotheses, every G-ideal of $\mathrm{Int}(D^n)$ lies over a maximal ideal of D ([2], proof of Prop. XI.3.7). (Recall that a G-ideal, or Goldman ideal, is a prime ideal that is not an intersection of prime ideals strictly containing it.)

Lemma 3.3. *Let D be a Noetherian Hilbert ring with $\dim(D) > 1$, such that every maximal ideal of finite index in D is of height 1. Then every maximal ideal of $\mathrm{Int}(D^n)$ lies either over (0) or over a maximal ideal in D.*

Proof. Let $\mathcal{P}$ be a prime ideal of $\mathrm{Int}(D^n)$ not lying over (0) or a maximal ideal of finite index in D. We will show that if $\mathcal{P}$ does not lie over a maximal ideal of $D[x_1,\ldots,x_n]$, then it is itself not maximal. Since Hilbert rings are characterized by the fact that every maximal ideal of $D[x_1,\ldots,x_n]$ lies over a maximal ideal of D, this does it. Suppose $\mathcal{P}' = \mathcal{P} \cap D[x_1,\ldots,x_n]$ is not maximal. Let $\mathcal{P}' \subset \mathcal{Q}'$ be a proper containment of prime ideals, and $Q = \mathcal{Q}' \cap D$. Then $[D : Q]$ is infinite.

We have $(D \setminus Q)^{-1}\mathrm{Int}(D^n) = D_Q[x_1,\ldots,x_n] = (D \setminus Q)^{-1}D[x_1,\ldots,x_n]$, so the prime ideals of $D_Q[x_1,\ldots,x_n]$ mediate a bijective correspondence between the prime ideals whose intersection with D is contained in Q of $\mathrm{Int}(D^n)$ on one hand and of $D[x_1,\ldots,x_n]$ on the other hand. Therefore $\mathcal{P}' \subset \mathcal{Q}'$ corresponds to a proper containment of prime ideals $\mathcal{P} \subset \mathcal{Q}$ in $\mathrm{Int}(D^n)$ and $\mathcal{P}$ is not maximal. □

The next lemma is only needed to show the redundancy of the d-ring property for domains of $\dim(D) > 1$ in Theorem 4.2. If we did not care about this redundancy, we could skip the lemma and show Theorem 3.5 (by the same proof) under the hypothesis "Noetherian d-ring and, if $\dim(D) > 1$, also Hilbert ring".

Lemma 3.4 ([2], proof of Prop. XI.3.7) *Let D be a Noetherian Hilbert ring with $\dim(D) > 1$, such that every maximal ideal of finite index is of height 1. Then no maximal ideal of $\mathrm{Int}(D^n)$ lies over (0) in D.*

Proof. Let $\mathcal{M}$ be a maximal ideal of $\mathrm{Int}(D^n)$ and suppose $\mathcal{M} \cap D = (0)$. Let Q be a prime ideal of D with $\mathrm{ht}(Q) > 1$. Then $[D : Q]$ is infinite and therefore $\mathrm{Int}(D^n) \subseteq D_Q[x_1,\ldots,x_n]$. By the correspondence between prime ideals of $D_Q[x_1,\ldots,x_n] = (D \setminus Q)^{-1}\mathrm{Int}(D^n)$ and those prime ideals $\mathcal{P}$ of $\mathrm{Int}(D^n)$ with $\mathcal{P} \cap D \subseteq Q$, $\mathcal{M} = \tilde{\mathcal{M}} \cap D$ for some maximal ideal $\tilde{\mathcal{M}}$ of $D_Q[x_1,\ldots,x_n]$. Now $\tilde{\mathcal{M}} \cap D_Q = (0)$, making (0) a G-ideal of D_Q ([10], proof of Thm. 31.8), but (0) never is a G-ideal in a Noetherian domain of dimension greater 1 ([13] Thm. 146), a contradiction. □

Theorem 3.5. *Let D be a Noetherian domain and either a d-ring of $\dim(D) \leq 1$ or a Hilbert ring of $\dim(D) > 1$. If for every maximal ideal M of D, either D/M is finite and $\mathrm{ht}(M) = 1$ or D/M is algebraically closed, then $\mathrm{Int}(D^n)$ has the Skolem property for all $n \in \mathbb{N}$.*

Proof. Given polynomials $f_1, \ldots, f_m \in \mathrm{Int}(D^n)$ that generate a proper ideal of $\mathrm{Int}(D^n)$, we wish to show that they are contained in a maximal ideal of the form $\mathcal{M}_{\underline{a},M} = \{f \in \mathrm{Int}(D^n) \mid f(\underline{a}) \in M\}$ for some $\underline{a} \in D^n$ and a maximal ideal M of D. Let $\mathcal{M}$ be a maximal ideal of $\mathrm{Int}(D^n)$ with $f_1, \ldots, f_m \in \mathcal{M}$ and let $P = \mathcal{M} \cap D$. By Lemma 3.3, P is either a maximal ideal of D or (0).

First case: $[D : P]$ finite. By Fact 2.2, $\mathcal{M} = \mathcal{M}_{\underline{b},\hat{P}}$, for some $\underline{b} \in \widehat{D_P}^n$. By continuity of the functions $f_i\colon \widehat{D_P}^n \to \widehat{D_P}$, there is a neighborhood U of $\underline{b}$, such that $f_i(U) \subseteq \hat{P}$ for $1 \leq i \leq m$. Pick $\underline{a} \in U \cap D^n$ (which exists since D^n is dense in $\widehat{D_P}^n$), then $f_i(\underline{a}) \in \hat{P} \cap D = P$, i.e. $f_i \in \mathcal{M}_{\underline{a},P}$ for $1 \leq i \leq m$.

Second case: $[D:P]$ infinite, $P \neq (0)$. $\mathrm{Int}(D^n) \subseteq D_P[x_1, \ldots, x_n]$, since $[D:P]$ is infinite. P is maximal, so by Fact 2.1, $\mathcal{M} = Q \cap \mathrm{Int}(D^n)$ for some maximal ideal Q of $D_P[x_1, \ldots, x_n]$ with $PD_P[x_1, \ldots, x_n] \subseteq Q$. As D_P/PD_P is algebraically closed, there exist $a_1, \ldots, a_n$ in D_P, such that $Q = (x_1 - a_1, \ldots, x_n - a_n) + PD_P[x_1, \ldots, x_n]$. As the a_i only matter mod PD_P and P is maximal, they can be chosen to lie in D, and we see that $\mathcal{M} = \mathcal{M}_{\underline{a},P}$ for some $\underline{a} = (a_1, \ldots, a_n) \in D^n$.

Third case: $P = (0)$. By the hypothesis of the theorem and Lemma 3.4, D is a d-ring in this case. By Fact 2.1 and the weak Nullstellensatz, $\mathcal{M}$ is exactly the kind of maximal ideal treated in Lemma 3.2, so we are done. $\square$

IV. *Nullstellensatz criterion for Noetherian domains*

We now have all the necessary components for the characterization (among all Noetherian domains) of those domains that satisfy the Nullstellensatz property for integer-valued polynomials in any number of variables, except that the necessity of part of the residue field condition hasn't been explained yet.

Remark 4.1 ([2] Prop. VII.4.2). If M is a finitely generated maximal ideal in a Skolem ring D, then D/M is either algebraically closed or finite.

Proof. Suppose otherwise. D/M not being algebraically closed, there exists a polynomial $f \in D[x]$, whose residue class in $(D/M)[x]$ is not a constant, that has no zero in D mod M. Also, D/M being infinite, $\mathrm{Int}(D) \subseteq D_M[x]$. By the Skolem property, $(M, f)\mathrm{Int}(D) = \mathrm{Int}(D)$, and therefore $(M, f)D_M[x] = D_M[x]$. This would make f a constant in $(D_M/MD_M)[x]$, contrary to assumption. $\square$

In the case of a Noetherian domain D with $\dim(D) > 1$ ([7] Thm. 2, [2] Prop. XI.3.7) and in the case of a Noetherian domain with $\dim(D) = 1$ and $\mathrm{char}(D) = 0$ ([7], Thm. 3, [2] Thm. XI.3.6), the following result is due to Chabert, several special cases (mostly concerning Dedekind domains) having previously been shown by Brizolis. Our approach also provides a new proof of the known cases.

If D is a field, the theorem below reduces to the statement that Hilbert's Nullstellensatz holds if and only if the weak Nullstellensatz holds, which in turn is the case if and only if the field is algebraically closed. Note, however, that we did not prove this, since we took the weak Nullstellensatz for algebraically closed fields for granted in the proof of Theorem 3.5.

Theorem 4.2. *Let D be a Noetherian domain. The following are equivalent:*

(1) *For all $n \in \mathbb{N}$, $\mathrm{Int}(D^n)$ has the stronger Nullstellensatz property.*

(2) *For all $n \in \mathbb{N}$, $\mathrm{Int}(D^n)$ has the Nullstellensatz property.*

(3) *For all $n \in \mathbb{N}$, $\mathrm{Int}(D^n)$ has the Skolem property.*

(4) *D is a Skolem ring.*

(5) *D is a d-ring and a Hilbert ring, and for every maximal ideal M of D, either D/M is finite and $\mathrm{ht}(M) = 1$ or D/M is algebraically closed.*

If $\dim(D) = 1$ then "Hilbert ring" in (5) is redundant.

If $\dim(D) > 1$ then "d-ring" in (5) is redundant.

Proof. The implications $(1 \Rightarrow 2)$, $(2 \Rightarrow 3)$ and $(3 \Rightarrow 4)$ are evident.

$(4 \Rightarrow 5)$ The d-ring property of D is just a special case of the Skolem property of $\mathrm{Int}(D)$ (Fact 1.2 (iii)), Skolem ring implies Hilbert ring by Remark 2.5, and the residue field condition is explained in Remarks 2.3 and 4.1.

$(5 \Rightarrow 3)$ is Theorem 3.5; and $(3 \Rightarrow 1)$ is a corollary of Theorem 2.4.

Finally, the redundancy of either d-ring or Hilbert ring, depending on the dimension of D, is evident from the hypothesis of Theorem 3.5. □

Since d-rings feature prominently in the above characterization, we should mention that they occur in abundance (cf. [2] §VII.2). For instance, $\mathbb{Z}$ is a d-ring, and so is every integral or finitely generated ring extension of a d-ring (Fact 1.3). In particular, every ring consisting of integers in a number field is a d-ring. Also, every ring R with $D[x] \subseteq R \subseteq K[x]$ for a domain D with quotient field K is a d-ring. No semi-local ring, however, is a d-ring (Fact 1.2 (ii)).

Remark 4.3. Another equivalent property that we could insert in Theorem 4.2 is

$(3\frac{1}{2})$ For all $n \geq 0$, $\mathrm{Int}(D^n)$ is a Skolem ring.

This is the statement that for all $n \geq 0$, $\mathrm{Int}(\mathrm{Int}(D^n))$, the ring of integer-valued polynomials in one indeterminate over $\mathrm{Int}(D^n)$, has the Skolem property. For a finite field, this and all other items of Theorem 4.2 are trivially false, so let D be an infinite domain. Obviously, $(3\frac{1}{2} \Rightarrow 4)$; to see $(3 \Rightarrow 3\frac{1}{2})$, note that the Skolem property of $\mathrm{Int}(D^{n+1})$ implies the Skolem property of $\mathrm{Int}(\mathrm{Int}(D^n))$. Indeed, under the customary identification of $\mathrm{Int}(\mathrm{Int}(D^n))$ with $\mathrm{Int}(D^{n+1})$ ([2] Prop. XI.1.1), the Skolem property of $\mathrm{Int}(\mathrm{Int}(D^n))$ says that every finitely generated proper ideal of this ring is contained in a maximal ideal of the form $\mathcal{M}_{a,\mathcal{Q}}$, with $a \in \mathrm{Int}(D^n)$ and $\mathcal{Q}$ a maximal ideal of $\mathrm{Int}(D^n)$, while the Skolem property of $\mathrm{Int}(D^{n+1})$ says the same thing with the additional requirements that $a \in D \subseteq \mathrm{Int}(D^n)$ and $\mathcal{Q}$ be of the form $\mathcal{Q} = \mathcal{M}_{\underline{b},M}$, with $\underline{b} \in D^n$ and M a maximal ideal of D.

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