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# INTEGER-VALUED POLYNOMIALS ON KRULL RINGS

Sophie Frisch

Abstract. If $R$ is a subring of a Krull ring $S$ such that $R_Q$ is a valuation ring for every finite index $Q = P \cap R$, $P$ in $\mathrm{Spec}^1(S)$, we construct polynomials that map $R$ into the maximal possible (for a monic polynomial of fixed degree) power of $PS_P$, for all $P$ in $\mathrm{Spec}^1(S)$ simultaneously. This gives a direct sum decomposition of $\mathrm{Int}(R, S)$, the $S$-module of polynomials with coefficients in the quotient field of $S$ that map $R$ into $S$, and a criterion when $\mathrm{Int}(R, S)$ has a regular basis (one consisting of 1 polynomial of each non-negative degree).

## Introduction

If $A$ is an infinite subset of a domain $S$, we write $\mathrm{Int}(A, S)$ for the $S$-module of polynomials with coefficients in the quotient field of $S$ that – when acting as a function by substitution of the variable – map $A$ into $S$. For $\mathrm{Int}(S, S)$, the ring of integer-valued polynomials on $S$, we write $\mathrm{Int}(S)$. Beyond the fact (known of old) that the binomial polynomials $\binom{x}{n} = \frac{x(x-1)\ldots(x-n+1)}{n!}$ form a basis of the free $\mathbb{Z}$-module $\mathrm{Int}(\mathbb{Z})$, the study of $\mathrm{Int}(S)$ originated with Pólya [16] and Ostrowski [15], who let $S$ be the ring of integers in a number field (their results have been generalized to Dedekind rings by Cahen [4]). $\mathrm{Int}(R, S)$ for $R \neq S$ has only begun to attract attention more recently [2, 3, 6, 8, 11, 13].

We will treat Pólya's and Ostrowski's questions in the case where $R \neq S$ and $S$ is a Krull ring; in particular the question when $\mathrm{Int}(R, S)$ is a free $S$-module that admits a regular basis, and the related one of determining the highest power of $PS_P$, where $P$ is a height 1 prime ideal of $S$, that a monic polynomial of fixed degree can map $R$ into. Following Pólya, we call a sequence of polynomials $(g_n)_{n\in\mathbb{N}_0}$ *regular*, if $\deg g_n = n$ for all $n$. One basic connection between a module of polynomials and the modules of leading coefficients should be kept in mind:

**0.1 Lemma.** *Let $R$ be a unitary subring of a field $K$, $M$ an $R$-submodule of $K[x]$, and $I_n = \{$ leading coefficients of $n$-th degree polynomials in $M\} \cup \{0\}$.*

*(i) If $(g_n)_{n\in\mathbb{N}_0}$ is a regular sequence of monic polynomials in $K[x]$ such that $I_n g_n \subseteq M$ for all $n$, then $M = \sum_{n=0}^{\infty} I_n g_n$ (direct sum).*

*(ii) A regular set of polynomials in $M$ is an $R$-basis if and only if the leading coefficient of the $n$-th degree polynomial generates $I_n$ as an $R$-module.*

*(iii) $M$ has a regular $R$-basis if and only if each $I_n$ is non-zero and cyclic.*

*Proof.* (i) If $(g_n)_{n\in\mathbb{N}_0}$ is as stated, then $\sum_{n=0}^{\infty} I_n g_n \subseteq M$ and the sum is direct, since $\deg(g_n) = n$ makes the $g_n$ linearly independent over $K$. An induction on $N = \deg f$

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shows that $f \in M$ implies $f \in \sum_{n=0}^{N} I_n g_n$. Indeed, for $N = 0$, $f \in I_0 = g_0 I_0$, and if $N > 0$ and $a_N$ is $f$'s leading coefficient, then $a_N \in I_N$, so $h = f - a_N g_N \in M$ and $h \in \sum_{n=0}^{N-1} I_n g_n$ by induction hypothesis. (ii) and (iii) are easy. □

## 1. Polynomials mapping a Set into a Discrete Valuation Ring

Throughout section one, $v$ is a discrete valuation on a field $K$ with value-group $\Gamma_v = \mathbb{Z}$ and $v(0) = \infty$, and $R_v$ its valuation ring with maximal ideal $M_v$. In a kind of generic local regular basis theorem, we will establish the connection (well-known in special cases) between $\mathrm{Int}(A, R_v)$ and the maximal power of $M_v$ that a monic polynomial of degree $n$ can map $A$ into, for all $A \subseteq K$ for which this maximum exists for every $n$. A subset $A$ of the quotient field of a domain $R$ is called $R$-fractional if there exists a $d \in R \setminus \{0\}$ such that $dA \subseteq R$.

**1.0 Lemma.** *If $R$ is an integrally closed domain with quotient field $L$, $A \subseteq L$ and $f$ non-constant $\in L[x]$ then $f(A)$ is $R$-fractional if and only if $A$ is.*

*Proof.* Let $f \in L[x]$, $\deg f = n > 0$. If $f(A)$ is $R$-fractional there is a non-zero $d \in R$, with $df(a) \in R$ for every $a \in A$. Let $c \in R \setminus \{0\}$, such that $cf \in R[x]$, and set $g = cdf = c_n x^n + \ldots + c_0$. For every $a \in A$, $g(a) \in R$ implies that $c_n a$ is integral over $R$, therefore $c_n a \in R$ and $c_n A \subseteq R$. The converse is clear. □

Since a set $B \subseteq K$ is $R_v$-fractional if and only if $\min_{b \in B} v(b)$ exists, Lemma 1.0 shows that $A$ being $R_v$-fractional is necessary and sufficient for $\min_{a \in A} v(f(a))$ to exist for any non-constant $f \in K[x]$. To exclude polynomials identically zero on $A$, for which $\min_{a \in A} v(f(a)) = \infty$, we need $\deg f < |A|$, so that the conditions on $A$ in Lemma 1.1 below are necessary.

**1.1 Lemma.** *Let $n \in \mathbb{N}_0$. If $A$ is an $R_v$-fractional subset of $K$ with $|A| > n$, then* $\max\{\min_{a \in A} v(f(a)) \mid f \text{ monic } \in K[x],\ \deg f = n\}$ *exists.*

*Proof.* The case $n = 0$ is trivial; so let $n > 0$ and $m \in \mathbb{N}$ such that $A$ is not contained in any union of $n$ cosets of $M_v^m$ in $K$. Such an $m$ exists, since $n < |A|$ and by the Krull Intersection Theorem $\bigcap_{m \in \mathbb{N}} M_v^m = (0)$. We show that for every monic $f \in K[x]$ of degree $n$ there exists an $a_0 \in A$ with $v(f(a_0)) < nm$ (and consequently $\max\{\min_{a \in A} v(f(a)) \mid f \text{ monic } \in K[x],\ \deg f = n\} < nm$).

Let $v'$ be an extension of $v$ to the splitting field of $f$ over $K$, $R_{v'}$ its valuation-ring with maximal ideal $M_{v'}$, and $e = [\Gamma_{v'} : \Gamma_v]$. $A$ is not contained in any union of $n$ cosets of ${M_{v'}}^{me}$ in $K'$. Pick an $a_0 \in A$ that is not in $u + {M_{v'}}^{me}$ for any root $u$ of $f$ in $K'$, then $v(f(a_0)) = v'(f(a_0)) = \sum_{i=1}^{n} v'(a_0 - u_i) < nm$. □

**1.2 Theorem.** *Let $A$ be an infinite, $R_v$-fractional subset of $K$. For $n \in \mathbb{N}_0$ set* $\gamma_{v,A}(n) = \max\{\min_{a \in A} v(f(a)) \mid f \text{ monic } \in K[x],\ \deg f = n\}$.

(i) ${M_v}^{-\gamma_{v,A}(n)} = \{$*leading coefficients of degree $n$ polynomials in* $\mathrm{Int}(A, R_v)\} \cup \{0\}$

(ii) *A regular basis of* $\mathrm{Int}(A, R_v)$ *is given by $(c_n g_n)_{n \in \mathbb{N}_0}$, with $g_n \in K[x]$ monic, $\deg g_n = n$, and $c_n \in K$, such that $\min_{a \in A} v(g_n(a)) = \gamma_{v,A}(n)$ and $v(c_n) = -\gamma_{v,A}(n)$.*

*Proof.* Let $I_{n,v} = \{$leading coefficients of degree n polynomials in $\mathrm{Int}(A, R_v)\} \cup \{0\}$. The leading coefficient $c_n$ of any n-th degree polynomial in $\mathrm{Int}(A, R_v)$ must satisfy $v(c_n) \geq -\gamma_{v,A}(n)$, so $I_{n,v} \subseteq {M_v}^{-\gamma_{v,A}(n)}$. Now, for $n \in \mathbb{N}_0$, let $g_n$ be monic of degree $n$ in $K[x]$ with $\min_{a \in A} v(g_n(a)) = \gamma_{v,A}(n)$ (such things exist by dint of Lemma 1.1) then ${M_v}^{-\gamma_{v,A}(n)} g_n \subseteq \mathrm{Int}(A, R_v)$, so ${M_v}^{-\gamma_{v,A}(n)} \subseteq I_{n,v}$. This shows (i) and also that

$I_{n,v} g_n \subseteq \mathrm{Int}(A, R_v)$ for all $n \in \mathbb{N}_0$. (ii) follows by Lemma 0.1 and the fact that ${M_v}^{-\gamma_{v,A}(n)} = c_n R_v$ for every $c_n \in K$ with $v(c_n) = -\gamma_{v,A}(n)$. □

Before deriving a formula for $\max\{\min_{a\in A} v(f(a)) \mid f \text{ monic} \in K[x], \deg f = n\}$, when $A$ is a subring of $R_v$, we check that the other plausible way of normalizing the polynomials would yield the same value. We also see that polynomials mapping $A \subseteq R_v$ into the maximal possible power of $M_v$ can be chosen to split with their roots in any set that $M_v$-adically approximates $A$ (for instance in $A$ itself, or, if $R_v$ is the localization of a ring $R$ at a prime ideal of finite index, in $R$). We need a lemma from [7] (but include the proof).

**1.3 Lemma.** *Let $f \in R_v[x]$, not all of whose coefficients lie in $M_v$, split over $K$, as $f(x) = d(x - b_1) \cdot \ldots \cdot (x - b_m) \cdot (x - c_1) \cdot \ldots \cdot (x - c_l)$ with $v(b_i) < 0$, $v(c_i) \geq 0$, and put $f_+(x) = (x - c_1) \cdot \ldots \cdot (x - c_l)$. Then for all $r \in R_v$ $\;v(f(r)) = v(f_+(r))$.*

*Proof.* For $r \in R_v$ $v(r - b_i) = v(b_i)$ and so $v(f(r)) = v(d) + \sum_{i=1}^{m} v(b_i) + v(f_+(r))$; we show $v(d) = -\sum_{j=1}^{m} v(b_i)$. Consider $d^{-1} f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$. Since $f \in R_v[x] \setminus M_v[x]$, $v(d) = -\min_{0\leq k\leq n} v(a_k)$. But $a_k$ is the elementary symmetric polynomial of degree $n - k$ in the $b_i$ and $c_i$, so the minimal valuation is attained by $v(a_{n-m}) = \sum_{i=1}^{m} v(b_i)$. □

**1.4 Proposition.** *Let $A \subseteq R_v$ and $0 \leq n < |A|$, then $\alpha$ and $\gamma$ below are equal:*

$$\alpha \;=\; \max\{\min_{a\in A} v(f(a)) \mid f \in R_v[x] \setminus M_v[x],\ \deg f = n\},$$
$$\gamma \;=\; \max\{\min_{a\in A} v(f(a)) \mid f \text{ monic } \in K[x],\ \deg f = n\}.$$

*If, furthermore, $B \subseteq R_v$, such that $B$ intersects every coset of ${M_v}^l$ that $A$ intersects, for all $l \in \mathbb{N}$, then $\delta$ below is equal to $\alpha$ and $\gamma$; and so is $\beta$, if $B$ is also a ring:*

$$\beta \;=\; \max\{\min_{a\in A} v(f(a)) \mid f \in B[x] \setminus (M_v \cap B)[x],\ \deg f = n\},$$
$$\delta \;=\; \max\{\min_{a\in A} v(f(a)) \mid f(x) = \textstyle\prod_{i=1}^{n}(x - d_i),\ d_i \in B\}.$$

*Proof.* Let $B$ be a fixed subset of $R_v$ that intersects every coset of every power of $M_v$ that $A$ intersects (e.g. $B = R_v$, when only interested in $\alpha$ and $\gamma$). For $n = 0$ all four expressions are equal to 0; now consider a fixed $n > 0$. Clearly $\delta \leq \gamma$ and, if $B$ is a ring, $\delta \leq \beta \leq \alpha$. Also $\gamma \leq \alpha$, because, given $f$ monic in $K[x]$, there exists a $d \in R_v$ such that $df = g \in R_v[x] \setminus M_v[x]$ and for all $a \in A$ $v(g(a)) = v(d) + v(f(a)) \geq v(f(a))$, and so $\min_{a\in A} v(g(a)) \geq \min_{a\in A} v(f(a))$.

To show $\alpha \leq \delta$, we fix $f \in R_v[x] \setminus M_v[x]$ of degree $n$ and construct a monic $g$ that splits with roots in $B$ such that $v(g(a)) \geq \min_{a\in A} v(f(a))$ for all $a \in A$. Let $v'$ be an extension of $v$ to the splitting field of $f$ over $K$. For all $a \in A$, $v'(f(a)) = v'(f_+(a))$ with $f_+(x) = \prod_{i=1}^{l}(x - c_i)$, where the $c_i$ are the roots of $f$ in $R_{v'}$, by Lemma 1.3. Put $s = \min_{a\in A} v'(f_+(a))$. We replace each $c_i$ by a $d_i \in B$ chosen such that $\prod_{i=1}^{l}(x - d_i) = h(x)$ satisfies: for all $a \in A$ $v'(h(a)) \geq s$. If $(c_i + M_{v'}^k) \cap A \neq \emptyset$ for all $k \in \mathbb{N}$, we pick $d_i$ out of $(c_i + M_{v'}^s) \cap B$; otherwise out of $(c_i + M_{v'}^k) \cap B$ with $k$ maximal such that $(c_i + M_{v'}^k) \cap A \neq \emptyset$. Since the intersection of a residue class of $M_{v'}^k$ in $R_{v'}$ with $R_v$ is either empty or an entire residue class of a power of $M_v$ in $R_v$, and $B$ intersects all of these that $A$ intersects, it is possible to find such $d_i$ in $B$. Now for every $a \in A$ either $v'(a - d_i) \geq v'(a - c_i)$ for all $i$ and so $v'(h(a)) \geq v'(f_+(a)) \geq s$, or $v'(a - d_i) \geq s$ for some $i$ and hence $v'(h(a)) \geq s$. To get a polynomial of degree $n$, set $g(x) = (x - d_0)^{n-l} h(x)$, $d_0 \in B$. □

## 2. Polynomials mapping into a maximal Power of $M_v$

If $R$ is an infinite subring of a discrete valuation ring $R_v$, we will construct polynomials $g_n(x) = (x - a_1)\dots(x - a_n)$ that map $R$ into the maximal possible (for a monic polynomial of degree $n$) power of $M_v$, by finding sequences $(a_i)$ in $R$ that show a nice distribution among the cosets of $M_v^n \cap R$, to serve as roots.

This generalizes a procedure of Pólya [16] (also used by Gunji and McQuillan [12, 14], Cahen [4] and others) for the special case where $R_v = R_Q$, $Q$ being a prime ideal of index $q$ in $R$ such that $R_Q$ is a discrete valuation ring: Pick $\pi \in Q \setminus Q^2$ and a complete set of residues $r_0, ..., r_{q-1}$ of $Q$ in $R$ and define $a_n = \sum_{i\geq 0} r_{c_i}\pi^i$, if $n = \sum_{i\geq 0} c_i q^i$ is the $q$-adic expansion of $n$. The resulting polynomials map $R$ into the highest possible power of $Q$ and can be used to give a regular basis of $\mathrm{Int}(R_v)$ (most clearly stated in [14]). Gilmer [10] has remarked that the construction even works for $\mathrm{Int}(D)$, $D$ a quasi-local ring with principal maximal ideal.

The $\mathcal{I}$-sequences below are defined for any commutative ring $R$. All sequences are indexed by an initial segment of $\mathbb{N}$ or $\mathbb{N}_0$. Quantifiers over indices of such a sequence are assumed to range over precisely the index-set.

**2.0 Definition.** For a set $\mathcal{I}$ of ideals in a commutative ring $R$ we define an *$\mathcal{I}$-sequence* in R to be a sequence $(a_n)$ of elements in R with the property

$$\forall I \in \mathcal{I} \quad \forall n, m \qquad a_n \equiv a_m \mod I \quad \Longleftrightarrow \quad [R : I] \mid n - m.$$

We define a *homogeneous* $\mathcal{I}$-sequence to be one with the additional property

$$\forall I \in \mathcal{I} \quad \forall n \geq 1 \qquad a_n \in I \iff [R : I] \mid n.$$

(Any infinite $[R : I]$ we regard as dividing 0, but no other integer.) Note that $a_1, a_2, ..$ is a homogeneous $\mathcal{I}$-sequence if and only if $0{=}a_0, a_1, a_2, ..$ is an $\mathcal{I}$-sequence.

**2.1 Proposition.** *Let $\mathcal{I} = \{I_n \mid n \in \mathbb{N}\}$ be a descending chain of ideals in a commutative ring $R$, then there exists an infinite homogeneous $\mathcal{I}$-sequence in $R$.*

*Proof.* Put $I_0 = R$. For $k \geq 0$, if $[I_k : I_{k+1}]$ is finite, let $\{a_j^{(k)} \mid 0 \leq j < [I_k : I_{k+1}]\}$ be a system of representatives of $I_k : I_{k+1}$ with $a_0^{(k)} = 0$, otherwise let $(a_j^{(k)})_{j\in\mathbb{N}_0}$ be a sequence in $I_k$ of elements pairwise incongruent mod $I_{k+1}$, with $a_0^{(k)} = 0$. If $I_N \in \mathcal{I}$ with $[R : I_N]$ finite, then every $n < [R : I_N]$ has a unique representation $n = \sum_{k=0}^{N-1} j_k [R : I_k]$ with $0 \leq j_k < [I_k : I_{k+1}]$, and we set $a_n = \sum_{k=0}^{N-1} a_{j_k}^{(k)}$. If the indices of ideals in $\mathcal{I}$ get arbitrarily large while remaining finite, this defines our $\mathcal{I}$-sequence inductively. Otherwise there exists $I_N \in \mathcal{I}$ of maximal finite index such that either $[I_N : I_{N+1}]$ is infinite or $I_m = I_N$ for $m \geq N$. Define $a_n$ for $n < [R : I_N]$ as above. Then, in the first case, set $a_m = a_q{}^{(N)} + a_r$ for $m = q\,[R : I_N] + r$ with $0 \leq r < [R : I_N]$, and $a_m = a_r$ in the second. $\square$

**2.2 Facts.**

(i) *For $I \in \mathcal{I}$ of finite index in $R$, any $[R : I]$ consecutive terms of an $\mathcal{I}$-sequence form a complete set of representatives of $R$ mod $I$.*

(ii) *If $(a_i)_{i=1}^n$ is an $\mathcal{I}$-sequence in $R$ then $(r - a_i)_{i=1}^n$ is an $\mathcal{I}$-sequence for every $r \in R$ and $(a_n - a_{n-i})_{i=0}^{n-1}$ is a homogeneous $\mathcal{I}$-sequence.*

The following lemma will be needed for globalization.

**2.3 Lemma.** *If $a_1, ..., a_l$ is an $\mathcal{I}$-sequence for a chain of ideals $\mathcal{I}$, $J \in \mathcal{I}$ with $[R : J] > l$, and $b_1, ..., b_l \in R$ such that $b_n \equiv a_n \bmod J$ for $1 \le n \le l$, then $(b_n)$ is also an $\mathcal{I}$-sequence, and homogeneous if $(a_n)$ is.*

*Proof.* Let $I \in \mathcal{I}$ and $1 \le n, m \le l$. First suppose $n \equiv m \bmod [R : I]$. Then $n = m$ or $[R : I] < l$. In the latter case $J \subseteq I$, so $b_n \equiv a_n \equiv a_m \equiv b_m \bmod I$. Now suppose $n \not\equiv m \bmod [R : I]$. Either $J \subseteq I$ or $I \subseteq J$. If $J \subseteq I$ then $b_n \equiv a_n \not\equiv a_m \equiv b_m \bmod I$. If $I \subseteq J$ then $b_n \equiv a_n \not\equiv a_m \equiv b_m \bmod J$ (because $0 \ne n - m < [R : J]$), hence $b_n \not\equiv b_m \bmod I$. Homogeneity is shown similarly. $\square$

From now on, $R$ is always an infinite subring of a discrete valuation ring $R_v$. Note that the definitions of $\alpha_{v,R}(n)$ and $v$-sequence below depend only on $M_v$ and $R$, and thus not distinguish between equivalent valuations.

**2.4 Definition.** A *$v$-sequence for $R$* is a $\{M_v^n \cap R \mid n \in \mathbb{N}\}$-sequence in $R$. In other words, $(a_n)$ is a $v$-sequence for $R$ if and only if for all $n \in \mathbb{N}$ and all $i$, $j$,

$$a_i - a_j \in {M_v}^n \quad\iff\quad [R : {M_v}^n \cap R] \mid i - j$$

and a *homogeneous* $v$-sequence if in addition, for all $n \in \mathbb{N}$ and all $j \ge 1$,

$$a_j \in {M_v}^n \quad\iff\quad [R : {M_v}^n \cap R] \mid j\,.$$

If $[R : {M_v}^n \cap R]$ is infinite, distinct elements of a $v$-sequence must be incongruent mod ${M_v}^n \cap R$. Proposition 2.1 guarantees the existence of an infinite homogeneous $v$-sequence for every infinite subring $R$ of every discrete valuation ring $R_v$.

**2.5 Definition.** For $n \in \mathbb{N}_0$, $R$ an infinite subring of $R_v$ and $q \in \mathbb{N}$, let

$$\alpha_{v,R}(n) = \sum_{j\ge1} \left[\frac{n}{[R : {M_v}^j \cap R]}\right] \qquad \text{and} \qquad \alpha_q(n) = \sum_{j\ge1} \left[\frac{n}{q^j}\right].$$

Infinite indices are allowed; $\frac{n}{\infty} = 0$. Since $R$ is infinite, $\alpha_{v,R}(n)$ is always a finite number. We will frequently use the fact that $\alpha_{v,R}(n) > 0$ if and only if $n \ge [R : M_v \cap R]$. If $Q$ is a prime ideal in a domain $D$, such that $D_Q$ is a discrete valuation ring, we write $v_Q$ for the corresponding valuation with value group $\mathbb{Z}$.

**2.6 Facts.**

(i) *If $Q$ is a prime ideal of finite index $q$ in $R$ such that $R_Q$ is a discrete valuation ring, then $\alpha_{v_Q,R}(n) = \alpha_q(n)$ for all $n$.*

(ii) *If $v$ is a discrete valuation, $R$ an infinite subring of $R_v$ and $v'$ an extension of $v$ with $[\Gamma_{v'} : \Gamma_v] = e$ finite, then $\alpha_{v',R}(n) = e \cdot \alpha_{v,R}(n)$ for all $n$.*

*Proof.* (i) Since $Q$ is maximal, $(QR_Q)^n \cap R = Q^n$ for all $n$. Using the fact that $Q$ contains a generator of $QR_Q$ one sees that $[R : Q^n] = [R_Q : (QR_Q)^n] = q^n$ for all $n$. (ii) For $k \in \mathbb{N}$, ${M_{v'}}^k \cap R = ({M_{v'}}^k \cap R_v) \cap R = {M_v}^{\lceil \frac{k}{e} \rceil} \cap R$, where $\lceil x \rceil$ denotes the smallest integer greater or equal $x$. Each number $\left[\frac{n}{[R:{M_v}^j \cap R]}\right]$ appears $e$ times, as $\left[\frac{n}{[R:{M_{v'}}^k \cap R]}\right]$ for $k = (j-1)e + 1, \ldots, je$, in the sum for $\alpha_{v',R}(n)$. $\square$

In the remainder of section two, $v$ is assumed to have value-group $\mathbb{Z}$.

**2.7 Lemma.** *Let $(a_i)_{i=1}^{n+1}$, $(b_i)_{i=1}^{n}$ and $(c_i)_{i=1}^{n}$ be $v$-sequences for $R$, and $(c_i)_{i=1}^{n}$ homogeneous, then*

(a) $v(c_1 \cdot \ldots \cdot c_n) = \alpha_{v,R}(n) \le v(b_1 \cdot \ldots \cdot b_n) \le \alpha_{v,R}(n) + \max_{1\le i\le n} v(b_i)$,

(b) $v\big(\prod_{i=1}^{n}(a_{n+1}-a_i)\big) = \alpha_{v,R}(n) \le v\big(\prod_{i=1}^{n}(r-b_i)\big)$ *for all $r \in R$.*

*Proof.* $v(c_1 \cdot \ldots \cdot c_n) = \sum_{j\ge 1} \big|\{i \mid 1 \le i \le n,\ v(c_i) \ge j\}\big|$ and similarly for the $b_i$. Since for finite index ${M_v}^j \cap R$ every $[R : {M_v}^j \cap R]$ successive terms of a $v$-sequence form a complete residue system of $R$ mod ${M_v}^j \cap R$, we have $\forall j \in \mathbb{N}$

$$\big|\{i \mid v(c_i) \ge j\}\big| = \left[\frac{n}{[R : {M_v}^j \cap R]}\right] \le \big|\{i \mid v(b_i) \ge j\}\big| \le \left[\frac{n}{[R : {M_v}^j \cap R]}\right] + 1$$

This implies (a) (and, since the 1 on the right can only occur if $[R : {M_v}^j \cap R] \not| n$, $v(b_1 \cdot \ldots \cdot b_n) \le \alpha_{v,R}(n) + \max_{1\le i\le n} v(b_i) - \max\{j \mid [R : {M_v}^j \cap R] \text{ divides } n\}$). By Fact 2.2 (ii) about $\mathcal{I}$-sequences, (b) is a special case of (a). □

**2.8 Theorem.** *Let $R$ be an infinite subring of $R_v$. An $R_v$-basis of* $\mathrm{Int}(R, R_v)$ *is given by*

$$f_0 = 1 \qquad \textit{and} \qquad f_n(x) = \frac{\prod_{i=1}^{n}(x-a_i)}{\prod_{i=1}^{n}(a_{n+1}-a_i)} \qquad (n \ge 1),$$

*where $(a_n)_{n=1}^{\infty}$ is a $v$-sequence for $R$.*

*Proof.* An infinite $v$-sequence $(a_n)_{n=1}^{\infty}$ in $R$ exists by Proposition 2.1 applied to $\{{M_v}^n \cap R \,|\, n \in \mathbb{N}\}$. The $f_n$, being a $K$-basis of $K[x]$, are free generators of the $R_v$-module they generate in $K[x]$, call this module $F$. Since by Lemma 2.7 every $f_n$ maps $R$ to $R_v$, $F \subseteq \mathrm{Int}(R, R_v)$. For the reverse inclusion we show the stronger statement that $\mathrm{Int}(A, R_v) \subseteq F$, where $A = \{a_n \,|\, n \in \mathbb{N}\}$. Let $f \in \mathrm{Int}(A, R_v)$, $f = \sum_{j=0}^{N} l_j f_j$ with $l_j \in K$. We show inductively that the $l_j$ are in $R_v$. $l_0 = f(a_1) \in R_v$. The induction hypothesis is $l_j \in R_v$ for $0 \le j < n$. Using this and the facts that $f_j(a_i) = 0$ for $j \ge i$ and $f_j(a_{j+1}) = 1$, we see that $f(a_{n+1}) = l_n + \sum_{j=0}^{n-1} l_j f_j(a_{n+1})$. Since $f_j(a_i) \in R_v$ for all $i$, $j$ (by Lemma 2.7) and $f \in \mathrm{Int}(A, R_v)$, the sum on the right as well as $f(a_{n+1})$ is in $R_v$, therefore $l_n \in R_v$. □

*Remark.* For an infinite subring $R$ of $R_v$ and $A \subseteq R$, the proof of Theorem 2.8 shows that if $A$ contains an infinite $v$-sequence for $R$, then $\mathrm{Int}(A, R_v) = \mathrm{Int}(R, R_v)$. The converse holds, too (the criterion for $\mathrm{Int}(A, R_v) = \mathrm{Int}(R, R_v)$ in [7] is easily seen to be equivalent to $A$ containing an infinite $v$-sequence for $R$).

**Corollary 1.** $\alpha_{v,R}(n) = \max\{\min_{r\in R} v(f(r)) \mid f$ *monic* $\in K[x],\ \deg f = n\}$ *and* $M_v^{-\alpha_{v,R}(n)} = \{$ *leading coefficients of $n$-th degree polynomials in* $\mathrm{Int}(R, R_v)\ \} \cup \{0\}$.

*Proof.* The second statement can be read off the theorem using Lemma 2.7 (b), the first one then follows by Theorem 1.2. □

Pólya's Satz IV [16] is a special case: if $P$ is a prime ideal in a domain $R$ such that $R_P$ is a discrete valuation ring and $[R : P] = q$, then (by Proposition 1.4 with $B = R$ and Fact 2.6 i) $\alpha_q(n) = \max\{\min_{r\in R} v_P(f(r)) \mid f \in R[x] \setminus P[x],\ \deg f = n\}$.

**Corollary 2.** *Let $g_n(x) = \prod_{i=1}^{n}(x - {a_i}^{(n)})$, where $({a_i}^{(n)})_{i=1}^{n}$ is a $v$-sequence for $R$ when $n \geq [R : M_v \cap R]$, and let $g_n$ be any monic polynomial in $R_v[x]$ of degree $n$ for $0 \leq n < [R : M_v \cap R]$. Further let, for $n \in \mathbb{N}_0$, $c_n \in K$ with $v(c_n) = -\alpha_{v,R}(n)$. Then $(c_n g_n)_{n\in\mathbb{N}_0}$ is an $R_v$-basis of* $\mathrm{Int}(R, R_v)$.

*Proof.* For all $n\in\mathbb{N}_0$, $r\in R$, $v(g_n(r))\geq\alpha_{v,R}(n)$ (by Lemma 2.7, when $n\geq[R:M_v\cap R]$, and because $g_n \in R_v[x]$ and $\alpha_{v,R}(n) = 0$ otherwise). By the maximality of $\alpha_{v,R}(n)$ (Corollary 1), $\min_{r\in R} v(g_n(r)) = \alpha_{v,R}(n)$. Therefore $(c_n g_n)_{n\in\mathbb{N}_0}$ is an $R_v$-basis of $\mathrm{Int}(R, R_v)$ by Corollary 1 and Theorem 1.2 (ii). □

## 3. Polynomials mapping a subring into a Krull Ring

*Notation.* Let $S$ be a domain with quotient field $K$, such that $S = \bigcap_{v\in\mathcal{V}} R_v$, $\mathcal{V}$ a set of discrete valuations (with value-group $\mathbb{Z}$) on $K$; and $R$ an infinite subring of $S$. We put $I_n = \{$leading coefficients of n-th degree polynomials in $\mathrm{Int}(R, S)\} \cup \{0\}$ and introduce names for recurring additional conditions:

(F) $\forall\, q \in \mathbb{N}\ \{Q \trianglelefteq R \mid [R : Q] = q \textit{ and } Q = M_v \cap R \textit{ for some } v \in \mathcal{V}\}$ is a finite set.

(C) For every prime ideal $Q$ of finite index in $R$, the set of ${M_v}^n \cap R$ with $n \in \mathbb{N}$, $v \in \mathcal{V}$, and $M_v \cap R = Q$, if not empty, forms a descending chain of ideals.

Note that (C) holds naturally in two cases: when there is only one $M_v$ such that $M_v \cap R = Q$, and when every ${M_v}^n \cap R$ with $M_v \cap R = Q$ is a power of $Q$.

**3.0 Lemma.** (Cahen [4]) *If $R$ is an infinite subring of a Krull ring $S$ and $q \in \mathbb{N}$, then $S$ has at most finitely many height 1 prime ideals $P$ with $[R : P \cap R] = q$.*

*Proof.* There exists $r \in R$ with $r^q - r \neq 0$. For every $P$ with $Q = R \cap P$ of index $q$ in $R$, $r^q - r \in Q \subseteq P$, so the statement follows by the definition of Krull ring. □

**3.1 Lemma.** *Let $v \in \mathcal{V}$ such that $M_v \cap R = Q \neq (0)$ and $L$ the quotient field of $R$. If $R_Q$ is a valuation ring, then it is a discrete valuation ring and $R_Q = R_v \cap L$. If $Q$ is also a maximal ideal then, for every $n \in \mathbb{N}$, $M_v^n \cap R$ is a power of $Q$.*

*Proof.* For any valuation ring $V$ with quotient field $L$ and maximal ideal $M$ we have $L\setminus V = \{r \in L^* \mid r^{-1} \in M\}$. Put $R_v \cap L = R_w$ and $M_v \cap L = M_w$, then $R_w$ and $R_Q$ are valuation rings with quotient field L and maximal ideals $M_w$ and $QR_Q$, respectively. $R \subseteq R_w$ and $M_w \cap R = M_v \cap R = Q$ imply $R_Q \subseteq R_w$ and also $QR_Q \subseteq M_w$. By the latter inclusion $L \setminus R_Q = \{r \in L^* \mid r^{-1} \in QR_Q\} \subseteq \{r \in L^* \mid r^{-1} \in M_w\} = L \setminus R_w$. This shows $R_Q = R_w = R_v \cap L$, so $R_Q$ is a discrete valuation ring and every $M_v^n \cap R_Q$ is a power of $QR_Q$. If $Q$ is maximal, then $(QR_Q)^k \cap R = Q^k$ for all $k$, so $M_v^n \cap R$ is a power of $Q$. □

**3.2 Lemma.** (C) *implies: For every finite set $\mathcal{M}$ of prime ideals of finite index in $R$ and every $m \in \mathbb{N}$, there exists a sequence $(a_i)_{i=0}^{m}$ in $R$ that is a homogeneous $v$-sequence for all $v$ in $\mathcal{V}$ with $M_v \cap R \in \mathcal{M}$, simultaneously.*

*Proof.* For every $Q \in \mathcal{M}$, $\mathcal{I}_Q = \{{M_v}^n \cap R \mid v \in \mathcal{V},\ n \in \mathbb{N},\ M_v \cap R = Q\}$ (if not empty) is a descending chain by (C), so there exists a homogeneous $\mathcal{I}_Q$-sequence $({a_i}^{(Q)})_{i=0}^{\infty}$ in $R$ by Proposition 2.1. For each $Q$ with $\mathcal{I}_Q \neq \emptyset$ let $I_Q$ be an element of $\mathcal{I}_Q$ with $[R : I_Q] > m$. $I_Q = {M_v}^n \cap R$ for some $v$ and $n$ and therefore contains $Q^n$. Since different $Q$ are co-prime, there exists, by the Chinese Remainder Theorem, a sequence $(a_i)_{i=0}^{m}$ in $R$ that is congruent to $({a_i}^{(Q)})_{i=0}^{m}$ modulo $I_Q$ for all $Q \in \mathcal{M}$. By Lemma 2.3, this a homogeneous $\mathcal{I}_Q$-sequence for all $Q \in \mathcal{M}$, i.e., a homogeneous $v$-sequence for all $v$ with $M_v \cap R \in \mathcal{M}$. □

From Lemma 3.0, Lemma 3.1 and the fact that the powers of an ideal $Q$ form a descending sequence, we conclude that the hypothesis of Theorem 3.4 below is satisfied in at least one natural setting:

**3.3 Fact.** *If $S$ is a Krull ring, $\mathcal{V}=\{v_P \mid P\in \mathrm{Spec}^1(S)\}$, and $R$ an infinite subring such that $R_Q$ is a valuation ring for every finite index $Q=P\cap R$, $P\in \mathrm{Spec}^1(S)$, then (C) and (F) both hold.*

In the following theorem, the case where $S$ is a Dedekind ring and $R=S$ is due to Cahen [4] (also pertinent: [5]).

**3.4 Theorem.** *Let $R$ be an infinite subring of $S=\bigcap_{v\in\mathcal{V}} R_v$. If* (C) *and* (F) *hold, then*

$$I_n = \bigcap\nolimits_{v\in\mathcal{V}} M_v^{-\alpha_{v,R}(n)} \qquad (n\in\mathbb{N}_0)$$

*and there exists a regular sequence of monic polynomials $(g_n)$ in $R[x]$ such that*

$$\mathrm{Int}(R,S)=\sum\nolimits_{n\geq 0} I_n g_n,$$

*namely, $g_n(x)=\prod_{i=1}^n (x-{a_i}^{(n)})$, where $({a_i}^{(n)})_{i=1}^n$ is a simultaneous $v$-sequence for all $v\in\mathcal{V}$ with $[R:M_v\cap R]\leq n$.*

*Proof.* $\mathrm{Int}(R,\bigcap_{v\in\mathcal{V}} R_v) = \bigcap_{v\in\mathcal{V}} \mathrm{Int}(R,R_v)$, therefore $I_n \subseteq \bigcap_{v\in\mathcal{V}} {M_v}^{-\alpha_{v,R}(n)}$ (by Theorem 2.8, Corollary 1). For the reverse inclusion, let $c\in\bigcap_{v\in\mathcal{V}} M_v^{-\alpha_{v,R}(n)}$. Set $\mathcal{V}_n=\{v\in\mathcal{V}\mid \alpha_{v,R}(n)>0\}=\{v\in\mathcal{V}\mid [R:M_v\cap R]\leq n\}$ then $\{M_v\cap R\mid v\in\mathcal{V}_n\}$ is finite by (F). Let $({a_i}^{(n)})_{i=1}^n$ in $R$ be a homogeneous $v$-sequence for all $v\in\mathcal{V}_n$ simultaneously (which exists by Lemma 3.2) and $g_n(x)=\prod_{i=1}^n(x-{a_i}^{(n)})$. Then $\min_{r\in R} v(g(r))\geq \alpha_{v,R}(n)$ for all $v\in\mathcal{V}$ (by Lemma 2.7 when $v\in\mathcal{V}_n$, and because $\alpha_{v,R}(n)=0$ and $g_n\in R[x]$ otherwise) which means $cg(x)\in\mathrm{Int}(R,\bigcap_{v\in\mathcal{V}} R_v)$ and hence $c\in I_n$. This completes the proof of the first statement and also shows, for all $n\geq 0$, that $I_n g_n\subseteq \mathrm{Int}(R,\bigcap_{v\in\mathcal{V}} R_v)$, so the second follows by Lemma 0.1. □

From now on, $S$ is a Krull ring. By convention, the empty intersection or product of ideals of $S$ equals $S$. We denote the set of height 1 prime ideals of $S$ by $\mathrm{Spec}^1(S)$ or $\mathcal{P}$. If $P\in\mathcal{P}$, we write $\alpha_{P,R}$ for $\alpha_{v_P,R}$ and, if $j\in\mathbb{N}_0$, $P^{(j)}$ for $(PS_P)^j\cap S$. With this notation we have, for $n\in\mathbb{N}_0$ and $P\in\mathcal{P}$:

$$\alpha_{P,R}(n)=\sum_{j\geq 1}\left[\frac{n}{[R:P^{(j)}\cap R]}\right].$$

**3.5 Lemma.** *Let $S$ be a Krull ring and $\mathcal{V}=\{v_P\mid P\in\mathcal{P}\}$. If* (C) *holds, then* $\mathrm{Int}(R,S)$ *has a regular basis* $\iff$ $\bigcap_{\substack{P\in\mathcal{P}\\ [R:P\cap R]\leq n}} P^{(\alpha_{P,R}(n))}$ *is principal for all $n$.*

*Proof.* $\alpha_{P,R}(n)\neq 0$ if and only if $[R:P\cap R]\leq n$. Since (F) holds by Lemma 3.0, this only happens for finitely many $P$ for each $n$. If $\{a_P\mid P\in\mathcal{P}\}$ is a set of integers, only finitely many of them non-zero, then $\bigcap_{P\in\mathcal{P}}(PS_P)^{-a_P}$ is principal if and only if $\bigcap_{P\in\mathcal{P}}(PS_P)^{a_P}$ is, namely if there exists $c\in K$ with $v_P(c)=a_P$ for all $P\in P$. If all $a_P$ are non-negative then $\bigcap_{P\in\mathcal{P}}(PS_P)^{a_P}=\bigcap_{a_P>0}P^{(a_P)}$. Applied to $\bigcap_{P\in\mathcal{P}}(PS_P)^{-\alpha_{P,R}(n)}$, which is $I_n$ by Theorem 3.4, with Lemma 0.1 (iii) in mind, this proves the claim. □

**3.6 Theorem.** *Let $R$ be an infinite subring of a Krull ring $S$, $\mathcal{P} = \mathrm{Spec}^1(S)$, $\mathcal{P}^* = \{P \in \mathcal{P} \mid [R : P \cap R]$ finite$\}$ and $\mathcal{Q} = \{R \cap P \mid P \in \mathcal{P}^*\}$. If $R_Q$ is a valuation ring for all $Q \in \mathcal{Q}$, then $R_Q$ is a discrete valuation ring for all $Q \in \mathcal{Q}$ and*

$$\mathrm{Int}(R,S) \textit{ has a regular basis } \iff \forall\, q \in \mathbb{N} \bigcap_{\substack{P\in\mathcal{P}\\ [R:R\cap P]=q}} P^{(e_P)} \textit{ is a principal ideal of } S,$$

*where $e_P$ is the ramification index of $PS_P$ over $QR_Q$, for $P \in \mathcal{P}^*$, $Q = P \cap R$.*

*Proof.* Let $\mathcal{P}_q = \{P \in \mathcal{P} \mid [R : P \cap R] = q\}$, $P \in \mathcal{P}_q$, $Q = P \cap R$, $L$ the quotient field of $R$, then by Lemma 3.1 $R_Q = S_P \cap L$ and $R_Q$ is a discrete valuation ring. $v'_P = (1/e_P)v_P$ is equivalent to $v_P$ and is an extension of $v_Q$ to $K$ with $[\Gamma_{v'_P} : \Gamma_{v_Q}] = e_P$. By the Facts 2.6 (ii) and (i), $\alpha_{P,R}(n) = \alpha_{v'_P,R}(n) = e_P\alpha_{Q,R}(n) = e_P\alpha_q(n)$.

If we call left and right side of the claimed equivalence (l) and (r), respectively, then (l) is equivalent to (l') '$\forall n \bigcap_{\substack{P\in\mathcal{P}\\ [R:P\cap R]\le n}} P^{(\alpha_{P,R}(n))}$ is principal' by Lemma 3.5 (whose condition (C) holds by Fact 3.3). We know that $\bigcap_{\substack{P\in\mathcal{P}\\ [R:P\cap R]\le n}} P^{(\alpha_{P,R}(n))} = \bigcap_{q\le n}\bigcap_{P\in\mathcal{P}_q} P^{(e_P\alpha_q(n))}$. The latter is clearly principal provided all $\bigcap_{P\in\mathcal{P}_q} P^{(e_P)}$ are; thus (r) $\Rightarrow$ (l').

For (l') $\Rightarrow$ (r), suppose $\bigcap_{q\le n}\bigcap_{P\in\mathcal{P}_q} P^{(e_P\alpha_q(n))} = s_nS$ for all $n$. We see that $s_qS = \bigcap_{P\in\mathcal{P}_q} P^{(e_P)} \cap \bigcap_{l<q}\bigcap_{P\in\mathcal{P}_l} P^{(e_P\alpha_l(q))}$, because $\alpha_q(q) = 1$. This allows an induction on $q$: from the formula for $s_qS$ we conclude that $\bigcap_{P\in\mathcal{P}_q} P^{(e_P)}$ is principal if $\bigcap_{P\in\mathcal{P}_l} P^{(e_P)}$ is principal for all $l < q$. □

**Corollary 1.** *If $R \subseteq S$ is an extension of Krull rings such that $\mathrm{ht}(P \cap R) \le 1$ for all height 1 prime ideals $P$ of $S$ then*

$$\mathrm{Int}(R,S) \textit{ has a regular basis } \iff \forall\, q \in \mathbb{N} \bigcap_{\substack{Q\in\mathrm{Spec}^1(R)\\ [R:Q]=q}} \mathrm{div}(QS) \textit{ is principal},$$

*where $\mathrm{div}(QS)$ means the smallest divisorial ideal containing $QS$.*

*Proof.* If $R \subseteq S$ is an extension of Krull rings with the stated property and $Q$ is in $\mathrm{Spec}^1(R)$, then $\mathrm{div}(QS) = \bigcap_{\substack{P\in\mathrm{Spec}^1(S)\\ P\cap R=Q}} P^{(e_P)}$, where $e_P = e(P|Q)$ is the ramification index of $PS_P$ over $QR_Q$ [1, p183]. □

In particular, if $R \subseteq S$ is an extension of Dedekind rings, then

$$\mathrm{Int}(R,S) \text{ has a regular basis } \iff \forall\, q \in \mathbb{N} \prod_{\substack{Q\in\mathrm{Spec}(R)\\ [R:Q]=q}} QS \text{ is principal.}$$

A different specialization gives Ostrowski's criterion [15]. If $S$ is a Krull ring,

$$\mathrm{Int}(S) \text{ has a regular basis } \iff \forall\, q \in \mathbb{N} \prod_{\substack{P\in\mathrm{Spec}^1(S)\\ [S:P]=q}} P \text{ is principal.}$$

When a regular basis exists, we can give a fairly explicit description of one. (For $\mathrm{Int}(S)$, $S$ a Dedekind ring, there also is a different construction by Gerboud [9].)

**Corollary 2.** *In the situation of Theorem 3.6, if* $\bigcap_{\substack{P\in\mathcal{P}\\ [R:P\cap R]=q}} P^{(e_P)} = c_q S$ $(q \in \mathbb{N})$ *then a regular basis of* $\mathrm{Int}(R,S)$ *is given by* $f_0 = 1$,

$$f_n(x) = \prod_{q\le n} c_q^{-\alpha_q(n)} \prod_{i=1}^{n} (x - a_i{}^{(n)}) \qquad (n \in \mathbb{N}),$$

*where* $(a_i{}^{(n)})_{i=1}^n \subseteq R$ *is a* $v_P$*-sequence for all* $P \in \mathcal{P}$ *with* $[R : P \cap R] \le n$.

*Proof.* $v_P(c_q^{-\alpha_q(n)}) = -e_P\alpha_q(n) = -\alpha_{P,R}(n)$ for the $P \in \mathcal{P}$ with $[R : P \cap R] = q$, and zero for all other $P \in \mathcal{P}$, so $v_P(\prod_{q\le n} c_q^{-\alpha_q(n)}) = -\alpha_{P,R}(n)$ for all $P \in \mathcal{P}$ (since $\alpha_{P,R}(n) = 0$ if $n < [R : P \cap R]$). Therefore the $f_n$ are an $S_P$-basis of $\mathrm{Int}(R, S_P)$ for all $P \in \mathcal{P}$ simultaneously, by Theorem 2.8, Corollary 2. □

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Institut für Mathematik C, Technische Universität Graz, Kopernikusgasse 24, A-8010 Graz, Austria
*e-mail:* `frisch@blah.math.tu-graz.ac.at`