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INTERPOLATION DOMAINS

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Abstract. Call a domain D with quotient field K an *interpolation domain* if, for each choice of distinct arguments $a_1, \dots, a_n$ and arbitrary values $c_1, \dots, c_n$ in D, there exists an integer-valued polynomial f (that is, $f \in K[X]$ with $f(D) \subseteq (D)$), such that $f(a_i) = c_i$ for $1 \leq i \leq n$. We characterize completely the interpolation domains if D is Noetherian or a Prüfer domain. In the first case, we show that D is an interpolation domain if and only if it is one-dimensional, locally unibranched with finite residue fields, in the second one, if and only if the ring $\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$ of integer-valued polynomials is itself a Prüfer domain. We also show that an interpolation domain must satisfy a double-boundedness condition, and thereby simplify a recent characterization of the domains D such that $\mathrm{Int}(D)$ is a Prüfer domain.

Introduction

Let K be a field. By the Lagrange interpolation formula, there is a polynomial, with coefficients in K, assigning given values to given distinct elements in K. The same does not hold for a domain D (which is not a field), as polynomials in $D[X]$ preserve congruences modulo every ideal of D. However, for certain domains D interpolation may be possible using *integer-valued polynomials*, that is, polynomials f with coefficients in the quotient field of D, such that $f(D) \subseteq D$. For instance, if $D = \mathbb{Z}$ is the ring of integers, it is clear that the binomial polynomials $\binom{X}{n} = \frac{X(X-1)\dots(X-n+1)}{n!}$ are integer-valued, hence, by linear combination, there is an integer-valued polynomial of degree n assigning given values to the integers 0 to n (and, changing X to $X - a$, to any finite set of consecutive integers).

It follows from a result of Carlitz [4, Theorem 7.1], together with Lagrange interpolation, that this is also the case for $D = \mathbb{F}_q[t]$ (a proof using an interpolation sequence that runs through $\mathbb{F}_q[t]$ bijectively has been given by Wagner [7]). Using such interpolation sequences one could extend this property to a discrete rank-one valuation domain with finite residue field (giving also an estimate for the degree of interpolation polynomials, [5]). More generally, it is known that interpolation by integer-valued polynomials is possible in every Dedekind domain with finite residue fields [5].

In this paper we completely classify the domains for which interpolation by integer-valued polynomials is possible, among Noetherian and Prüfer domains. We adopt the usual notation of $\mathrm{Int}(D)$ for the ring of integer-valued polynomials on the domain D, that is, $\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$, where K denotes the quotient field of D, and we set the following definition:

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Definition 1. The domain D is an *Interpolation Domain* if, for each choice of distinct arguments $a_1, \dots, a_n$ and arbitrary values $c_1, \dots, c_n$ in D, there exists $f \in \mathrm{Int}(D)$ such that $f(a_i) = c_i$, for $1 \leq i \leq n$.

In a first section, we give necessary conditions. We find that every interpolation domain is one-dimensional with finite residue fields and satisfies the following double-boundedness condition: for each nonzero $z \in D$ there is an integer n such that, for each maximal ideal $\mathfrak{m}$ containing z, $z \notin \mathfrak{m}^n$, and $|D/\mathfrak{m}| \leq n$.

In a second section, after giving some properties of localization, we characterize the interpolation domains in the Noetherian case: a one-dimensional Noetherian domain with finite residue fields is an interpolation domain if and only if it is locally unibranched (we recall the definition within the said section).

In the last section we let D be a Prüfer domain. We first note that an interpolation Prüfer domain is necessarily an almost Dedekind domain with finite residue fields. Under this condition, we show that the following conditions are equivalent: (i) D is an interpolation domain, (ii) D satisfies the double-boundedness condition, (iii) $\mathrm{Int}(D)$ is a Prüfer domain. The equivalence of (ii) and (iii) simplifies Alan Loper's characterization of the domains D such that $\mathrm{Int}(D)$ is a Prüfer domain [6].

As seen at the beginning, the case of a field is trivial by the Lagrange interpolation formula. In what follows, we always assume D to be a domain that is not a field.

1. Necessary conditions

We first note that the interpolation property amounts to the possibility of assigning arbitrary values to every pair of distinct elements.

Proposition 1.1. *The following assertions are equivalent for a domain D.*

(i) *D is an interpolation domain,*
(ii) *for each pair of distinct elements a, b in D, there exists $f \in \mathrm{Int}(D)$ such that $f(a) = 0$ and $f(b) = 1$,*
(iii) *for each pair of distinct elements a, b in D, and for each maximal ideal $\mathfrak{m}$ of D, there exists $f \in \mathrm{Int}(D)$ such that $f(a) \in \mathfrak{m}$ and $f(b) \notin \mathfrak{m}$.*

Proof. (i) $\Rightarrow$ (iii) Obvious.
(iii) $\Rightarrow$ (ii) Let $\mathfrak{A}$ be the set of polynomials $g \in \mathrm{Int}(D)$ such that $g(a) = 0$. Then $\mathfrak{A}$ is an ideal of $\mathrm{Int}(D)$. Let $\mathfrak{A}(b) = \{g(b) \mid g \in \mathfrak{A}\}$. Then $\mathfrak{A}(b)$ is an ideal of D. Assuming (iii), for each maximal ideal $\mathfrak{m}$, there is $f \in \mathrm{Int}(D)$ such that $f(a) \in \mathfrak{m}$ and $f(b) \notin \mathfrak{m}$. Then $g = f - f(a)$ belongs to $\mathfrak{A}$ and is such that $g(b) \notin \mathfrak{m}$. Hence $\mathfrak{A}(b)$ is not contained in any maximal ideal of D. Therefore $\mathfrak{A}(b) = D$, and in particular, there exists $f \in \mathfrak{A}$ such that $f(b) = 1$.
(ii) $\Rightarrow$ (i) Use linear combinations of products of the polynomials existing by (ii). □

Recall that, for each element $a \in D$, and each maximal ideal $\mathfrak{m}$ of D, the set $\mathfrak{M}_a = \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$ is a maximal ideal of $\mathrm{Int}(D)$ (with residue field isomorphic to $D/\mathfrak{m}$). The third condition of the Proposition 1.1 is clearly equivalent to saying that these ideals are distinct, for distinct elements of D. This simple remark is the key for our characterization of Noetherian interpolation domains [§2]. Hence, although it is just a rewording of the previous result, we give it as a corollary.

Corollary 1.2. *A domain D is an interpolation domain if and only if, for each pair of distinct elements a, b in D, and for each maximal ideal $\mathfrak{m}$ of D, the ideals $\mathfrak{M}_a$ and $\mathfrak{M}_b$ of $\mathrm{Int}(D)$ are distinct.*

Remark 1.3. From Proposition 1.1, it is also easy to generalize interpolation to several indeterminates: if D is an interpolation domain, there is an integer-valued polynomial in n indeterminates assigning given values to distinct arguments in D^n. Given a pair of distinct elements $\underline{a}, \underline{b}$ in D^n, it is enough to find a polynomial such that $f(\underline{a}) = 0$ and $f(\underline{b}) = 1$. Writing $\underline{a} = (\alpha_1, \ldots, \alpha_n)$ and $\underline{b} = (\beta_1, \ldots, \beta_n)$, then $\alpha_i \neq \beta_i$ for some i, hence there is an integer-valued polynomial $f(X_i)$ in one indeterminate, such that $f(\alpha_i) = 0$ and $f(\beta_i) = 1$. Such a polynomial can be considered as an integer-valued polynomial in n indeterminates.

Next we address the issue of finding interpolating polynomials of prescribed degree. For this, we first establish a property of continuity.

Proposition 1.4. *Let D be a domain, $\mathfrak{a}$ be an ideal, and $a \in D$. If $f \in \mathrm{Int}(D)$ is of degree $d \leq n$, then $z \in \mathfrak{a}^n$ implies $\big(f(a+z) - f(a)\big) \in \mathfrak{a}$.*

Proof. It suffices to prove the assertion for z of the special form $z = ty$, with $t \in \mathfrak{a}$ and $y \in \mathfrak{a}^{n-1}$. Indeed, each $z \in \mathfrak{a}^n$ is a finite sum $z = \sum_{i=1}^{r} z_i$ of such special elements, hence $\big(f(a+z) - f(a)\big)$ is a finite sum of elements of the form $\big(f(b_i + z_i) - f(b_i)\big)$. Replacing $f(X)$ by $f(X+a) - f(a)$, we assume that $f(0) = 0$, and show that $f(ty) \in \mathfrak{a}$. The result is obvious if $n = 0$. For $n \geq 1$, we let $g(X) = f(tX) - t^n f(X)$. Then g is a polynomial of degree at most $n-1$, and such that $g(0) = 0$. By induction, we have $g(y) \in \mathfrak{a}$. The result follows, since $f(ty) = g(y) + t^n f(y)$ and $t^n \in \mathfrak{a}$. □

In particular, for every ideal $\mathfrak{a}$ of D, each $f \in \mathrm{Int}(D)$ is a uniformly continuous function from D to D, in the $\mathfrak{a}$-adic topology: if $\deg(f) \leq n$, then $z \in \mathfrak{a}^{nh}$ implies $\big(f(a+z) - f(a)\big) \in \mathfrak{a}^h$.

Remark 1.5. In the case of a maximal ideal $\mathfrak{m}$, we can improve this result: if $\deg(f) \leq n$, then $z \in \mathfrak{m}^{n+h+1}$ implies $\big(f(a+z) - f(a)\big) \in \mathfrak{m}^h$. As above, we assume that $f(0) = 0$, and prove by induction on n that $z = yt$, with $y \in \mathfrak{m}^{n+h}$ and $t \in \mathfrak{m}$, implies $f(z) \in \mathfrak{m}^h$. As above, from $f(z) = g(y) - t^n f(z)$ and the induction hypothesis, it follows that $(1 + t^n) f(z) \in \mathfrak{m}^h$. Since $t \in \mathfrak{m}$, $f(z) \in \mathfrak{m}^h D_{\mathfrak{m}} \cap D$, that is, $f(z) \in \mathfrak{m}^h$.

We are ready to give two necessary conditions for the existence of an interpolating polynomial of given degree.

Lemma 1.6. *Let D be a domain and z be a non-zero element of D for which there exists a polynomial $f \in \mathrm{Int}(D)$ of degree n with $f(0) = 0$ and $f(z) = 1$. Then, for every prime ideal $\mathfrak{p}$ of D, we have*

(1) $z \notin \mathfrak{p}^n$,
(2) *if* $z \in \mathfrak{p}$, *then* $|D/\mathfrak{p}| \leq n$.

Proof. Proof. 1. Since we have $\big(f(z) - f(0)\big) \notin \mathfrak{p}$, it follows from Proposition 1.4 that $z \notin \mathfrak{p}^n$.
2. Assume that $n < |D/\mathfrak{p}|$. Choosing $n+1$ elements mutually incongruent modulo $\mathfrak{p}$ in D, and writing f as the Lagrange interpolation polynomial assigning the same values as f to these elements, we see that $f \in D_{\mathfrak{p}}[X]$ (see also [1, Corollary I.3.3]). Therefore $1 = f(z) - f(0) \in z D_{\mathfrak{p}}$. We reach a contradiction if z belongs to $\mathfrak{p}$. □

If z is a nonzero element and $\mathfrak{m}$ a maximal ideal of D, we denote by $w_{\mathfrak{m}}(z)$ the order of z with respect to $\mathfrak{m}$ (that is, the largest integer n such that $z \in \mathfrak{m}^n$). For an interpolation domain, this order is well defined:

Proposition 1.7. *Let D be an interpolation domain. Then D is one-dimensional with finite residue fields. Moreover, for each nonzero element $z \in D$, there is an integer n such that, for each maximal ideal $\mathfrak{m}$ containing z, we have the double boundedness condition $|D/\mathfrak{m}| \leq n$ and $w_{\mathfrak{m}}(z) \leq n$.*

Proof. Assume D is an interpolation domain. For each nonzero element $z \in D$, there exists a polynomial $f \in \mathrm{Int}(D)$, of some degree n, such that $f(0) = 0$, and $f(z) = 1$. It follows from the previous lemma that, for each prime ideal $\mathfrak{p}$ containing z, $z \notin \mathfrak{p}^n$ and $|D/\mathfrak{p}| \leq n$. In particular, each nonzero prime ideal has a finite residue field (since it contains a nonzero element), and D is one-dimensional. □

Remarks 1.8. (1) The proof of Lemma 1.6 suggests that, if D is an interpolation domain, then $\mathrm{Int}(D)$ is not contained in $D_{\mathfrak{p}}[X]$ for any prime ideal $\mathfrak{p}$. In fact, $\mathrm{Int}(D)$ is not contained in $B[X]$ for any overring B of D (that is, any ring B containing D and strictly contained in K). Indeed, consider a nonzero prime ideal $\mathfrak{P}$ of B. Then $\mathfrak{p} = \mathfrak{P} \cap D$ is a nonzero prime ideal of D. If $\mathrm{Int}(D) \subseteq B[X]$, then for each $f \in \mathrm{Int}(D)$, if a and b are congruent modulo $\mathfrak{p}$ in D, we have $\big(f(b) - f(a)\big) \in \mathfrak{P} \cap D = \mathfrak{p}$.
(2) If D is a one-dimensional local Noetherian domain with finite residue field, the conditions of Proposition 1.7 are satisfied. The characterization we give in the Noetherian case will show that these conditions are not sufficient [§2].
(3) The double-boundedness condition of Proposition 1.7 is very similar to the condition given by Alan Loper for the characterization of the domains D such that $\mathrm{Int}(D)$ is a Prüfer domain [6]. We come back to this issue in the last section.

2. Noetherian interpolation domains

We open this section with a property of localization. Recall that, for a multiplicative subset S of a domain D, we have the containment $S^{-1}\,\mathrm{Int}(D) \subseteq \mathrm{Int}(S^{-1}D)$ [1, Proposition I.2.2]. In general, this containment is strict, but if D is Noetherian, we have an equality [1, Theorem I.2.3].

Lemma 2.1. *Let D be an interpolation domain. Then $S^{-1}D$ is an interpolation domain for each multiplicative subset S of D.*

Proof. Let a/s, b/s be two distinct elements of $S^{-1}D$, where $a, b \in D$ and $s \in S$. By hypothesis, there exists a polynomial $f \in \mathrm{Int}(D)$ such that $f(a) = 0$ and $f(b) = 1$. Set $g(X) = f(sX)$, then $g\,(a/s) = 0$, $g\,(b/s) = 1$, and $g \in \mathrm{Int}(D) \subseteq \mathrm{Int}(S^{-1}D)$. The conclusion follows from Proposition 1.1. □

We turn now to the Noetherian case.

Proposition 2.2. *Let D be Noetherian. Then D is an interpolation domain if and only if $D_{\mathfrak{m}}$ is an interpolation domain for every maximal ideal $\mathfrak{m}$ of D.*

Proof. It remains to show that, if each $D_{\mathfrak{m}}$ is an interpolation domain, then so is D. Let $a \neq b$ in D, and $\mathfrak{m}$ be a maximal ideal of D. By hypothesis, there is a polynomial $f \in \mathrm{Int}(D_{\mathfrak{m}})$ such that $f(a) = 0$ and $f(b) = 1$. Since D is Noetherian, we have $\mathrm{Int}(D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}})$: there is $s \in D, s \notin \mathfrak{m}$ such that $sf \in \mathrm{Int}(D)$. Clearly $sf(a) \in \mathfrak{m}$ and $sf(b) \notin \mathfrak{m}$. The conclusion follows from Proposition 1.1. □

Remark 2.3. For non-Noetherian D, it may be that each $D_{\mathfrak{m}}$ is an interpolation domain while D is not. For instance, if D is an almost Dedekind domain with finite residue fields, then each $D_{\mathfrak{m}}$ is a discrete valuation domain with finite residue field, hence an interpolation domain (as noted in the introduction, see also Theorem 2.4 below), but D does not necessarily satisfy the double-boundedness condition of Proposition 1.7 (see §3 below).

By Proposition 1.7 we may assume that D is a one-dimensional domain with finite residue fields and by Proposition 2.2 that D is local with maximal ideal $\mathfrak{m}$. Under these hypotheses, it is known that the ideals $\mathfrak{M}_a = \{f \in \mathrm{Int}(D) \mid f(a) \in \mathfrak{m}\}$ are distinct if and only if D is *unibranched*, that is, the integral closure D' of D is a local ring (or equivalently a rank-one discrete valuation domain); and that, if D is not unibranched, there are only finitely many distinct maximal ideals of the form $\mathfrak{M}_a$ [1, Theorem V.3.1 & Proposition V.3.10]. ¿From Corollary 1.2 it then follows that D is an interpolation domain if and only it is unibranched. In the global case, we derive immediately the following characterization.

Theorem 2.4. *Let D be a Noetherian domain. Then D is an interpolation domain if and only if it is one-dimensional with finite residue fields and locally unibranched.*

Remarks 2.5. (1) Assume that D is a local one-dimensional Noetherian domain, with maximal ideal $\mathfrak{m}$ and finite residue field. Denote by $\widehat{D}$ the completion of D in the $\mathfrak{m}$-adic topology. Integer-valued polynomials are uniformly continuous functions and can be extended to $\widehat{D}$. Given $a \neq b$, there is clearly a continuous function φ such that $\varphi(a) = 0$ and $\varphi(b) = 1$. If the continuous functions from $\widehat{D}$ to $\widehat{D}$ were arbitrarily uniformly approximated by integer-valued polynomials, analogously to the classical Stone-Weierstrass theorem, it would be easy to find $f \in \mathrm{Int}(D)$ such that $f(a) = 0$ and $f(b) = 1$. This approximation property holds if only if D is *analytically irreducible* [1, Theorem III.5.3] (analytically irreducible means that $\widehat{D}$ is a domain and implies that D is unibranched).
(2) If D is a one-dimensional Noetherian local ring with finite residue field, either D is an interpolation domain, and the ideals $\mathfrak{M}_a$ are all distinct, or D is not unibranched, and there are only finitely many ideals of the form $\mathfrak{M}_a$ [1, Proposition V.3.10]. In general, for a quasi-local domain D, it may be that there are infinitely many ideals of the form $\mathfrak{M}_a$ which are not all distinct. This is obviously the case if $|D/\mathfrak{m}|$ is infinite, in which case $\mathrm{Int}(D) = D[X]$ is not an interpolation domain, while $\mathfrak{M}_a = \mathfrak{M}_b$ if and only if $a \equiv b \pmod{\mathfrak{m}}$. But here is a less trivial example. Let V be a valuation domain, with maximal ideal $\mathfrak{m}$, such that $V/\mathfrak{m}$ is finite and $\mathfrak{m}$ is a principal ideal. If the dimension of V is strictly greater than one, then V is not an interpolation domain. However, letting $\mathfrak{p} = \bigcap_1^\infty \mathfrak{m}^n$, then $\mathfrak{p} \neq (0)$ and $\mathfrak{M}_a = \mathfrak{M}_b$ in $\mathrm{Int}(V)$ if and only if $a \equiv b \pmod{\mathfrak{p}}$ [2, Théorème 2.2].

Corollary 2.6. *If D is a Noetherian interpolation domain, then each overring of D is an interpolation domain.*

Proof. Let B be an overring of the Noetherian interpolation domain D. It follows from Theorem 2.4 and from the Krull-Akizuki theorem that B is a one-dimensional Noetherian domain with finite residue fields. Moreover, for each maximal ideal $\mathfrak{n}$ of B, $B_{\mathfrak{n}}$ contains $D_{\mathfrak{m}}$ where $\mathfrak{m} = \mathfrak{n} \cap D$. Hence the integral closure $B'_{\mathfrak{n}}$ of $B_{\mathfrak{n}}$ is an overring of the integral closure $D'_{\mathfrak{m}}$ of $D_{\mathfrak{m}}$. Since $D'_{\mathfrak{m}}$ is a valuation domain, then so is $B'_{\mathfrak{n}}$. □

3. Prüfer Domains

We finally turn to the case where D is a Prüfer domain: For each maximal ideal $\mathfrak{m}$, $D_{\mathfrak{m}}$ is a valuation domain. If D satisfies the double-boundedness condition of Proposition 1.7, $D_{\mathfrak{m}}$ is a rank-one discrete valuation domain with finite residue field. In other words, D is an an almost Dedekind domain with finite residue fields. We derive the following characterization.

Theorem 3.1. *Let D be a Prüfer domain. Then D is an interpolation domain if and only if, for each nonzero element $z \in D$, there is an integer n such that, for each maximal ideal $\mathfrak{m}$ containing z, we have the double-boundedness condition $|D/\mathfrak{m}| \le n$ and $w_{\mathfrak{m}}(z) \le n$.*

Proof. It follows from Proposition 1.7 that the double boundedness condition is necessary. Conversely, as we said above, it implies that D is an almost Dedekind domain with finite residue fields. To each maximal ideal $\mathfrak{m}$ of D corresponds an essential valuation $v_{\mathfrak{m}}$ of D, and for $z \in D$, the order $w_{\mathfrak{m}}(z)$ is simply $v_{\mathfrak{m}}(z)$. Consider then $a \neq b$ in D and denote by $\mathcal{V}$ the set of essential valuations of D such that $v(b-a) > 0$. We shall construct a polynomial $f \in \mathrm{Int}(D)$ such that $f(a) = 0$ and $v\big(f(b)\big) = 0$ for each $v \in \mathcal{V}$. From Proposition 1.1 this will complete the proof since, on the other hand, $g = X - a$ is such that $g(a) = 0$ and $v\big(g(b)\big) = 0$ for each essential valuation $v \notin \mathcal{V}$. Since there is an integer n such that $v_{\mathfrak{m}}(b-a) \le n$ for each $\mathfrak{m}$ containing $(b-a)$, there is an integer e such that $ev(y)$ is a multiple of $v(b-a)$ for each $y \in D$ and each $v \in \mathcal{V}$ (for instance, $e = \mathrm{lcm}\{v(b-a) \mid v \in \mathcal{V}\}$). Since there is an integer n such that $|D/\mathfrak{m}| \le n$ for each $\mathfrak{m}$ containing $(b-a)$, there is also an integer q such that $v(y^{q-1} - 1) > 0$ for each $v \in \mathcal{V}$ and each $y \in D$ such that $v(y) = 0$ (for instance, $q - 1 = \mathrm{lcm}\{q_v - 1 \mid v \in \mathcal{V}\}$, where $q_v = |D/\mathfrak{m}_v|$). We define a sequence of polynomials by

$$f_0(X) = (X-a)^e, \text{ and } f_n(X) = \frac{f_{n-1}(X)\big(f_{n-1}(X)^{q-1} - 1\big)^e}{b-a}.$$

We see by induction that $v\big(f_n(y)\big)$ is a non-negative multiple of $v(b-a)$ for each $y \in D$ and each $v \in \mathcal{V}$, and hence, that $f_n \in \mathrm{Int}(D)$. On the other hand, we clearly have $f_n(a) = 0$ for all n. Finally, if $v\big(f_{n-1}(b)\big) > 0$, we have $v\big([f_{n-1}^{q-1}(b) - 1]^e\big) = 0$, hence $v\big(f_n(b)\big) = v\big(f_{n-1}(b)\big) - v(b-a)$. For $n \le e$, we then have

$$v\big(f_n(b)\big) = v\big(f_0(b)\big) - nv(b-a) = ev(b-a) - nv(b-a).$$

We conclude that f_e is the polynomial we are looking for. □

If $\mathrm{Int}(D)$ is a Prüfer domain, then D is an almost Dedekind domain with finite residue fields [1, Proposition VI.1.5]. On the other hand, the double-boundedness condition is very similar to the condition given by Alan Loper to characterize the almost Dedekind domains D such that $\mathrm{Int}(D)$ is a Prüfer domain [6]. Let us state Alan Loper's condition (as he did himself) in the case where the characteristic of D is 0: $\mathrm{Int}(D)$ is a Prüfer domain if and only if, for each prime element p of $\mathbb{Z}$, there is an integer n, such that, for each essential valuation v of D such that $v(p) > 0$, $v(p) \le n$ and $|D/\mathfrak{m}| \le n$. (In the case where the characteristic of D is a nonzero prime number q, he then gives a similar condition replacing $\mathbb{Z}$ by the ring of polynomials $\mathbb{F}_q[t]$.) Our condition is a priori stronger than Alan Loper's, since we do not restrict ourselves to the prime elements of a subring of D. In fact, it follows from the next result that both conditions are equivalent.

Corollary 3.2. *Let D be a Prüfer domain. Then D is an interpolation domain if and only if* $\mathrm{Int}(D)$ *is a Prüfer domain.*

Proof. If $\mathrm{Int}(D)$ is a Prüfer domain, then D is an almost Dedekind domain with finite residue fields [1, Proposition VI.1.5]. Let $\mathfrak{m}$ be a maximal ideal of D, b be an element of D, and consider the maximal ideal $\mathfrak{M}_b = \{h \in \mathrm{Int}(D) \mid h(b) \in \mathfrak{m}\}$ of $\mathrm{Int}(D)$. Clearly, the localization $\mathrm{Int}(D)_{\mathfrak{M}_b}$ is contained in the valuation domain $W = \{\varphi \in K(X) \mid \varphi(b) \in D_\mathfrak{m}\}$. Since $\mathrm{Int}(D)$ is a Prüfer domain, $\mathrm{Int}(D)_{\mathfrak{M}_b}$ is itself a valuation domain, and we have the equality $\mathrm{Int}(D)_{\mathfrak{M}_b} = W$ [3, Remark 2.5]. Since $D_\mathfrak{m}$ is a rank-one discrete valuation domain with finite residue field, it is an interpolation domain. Hence, for $a \neq b$, there exists a polynomial $f \in \mathrm{Int}(D_\mathfrak{m})$ such that $f(a) = 0$ and $f(b) = 1$. Obviously $\mathrm{Int}(D_\mathfrak{m}) \subseteq W$, thus $f \in \mathrm{Int}(D)_{\mathfrak{M}_b}$, and there is $g \in \mathrm{Int}(D)$, $g \notin \mathfrak{M}_b$ such that $gf \in \mathrm{Int}(D)$. Clearly $gf(a) \in \mathfrak{m}$ and $gf(b) \notin \mathfrak{m}$. It follows from Proposition 1.1 that D is an interpolation domain.

Conversely, if D is an interpolation domain, it satisfies the double-boundedness condition, and it is known that this condition, under Alan Loper's weaker form, implies that $\mathrm{Int}(D)$ is a Prüfer domain [1, Proposition VI.4.4 and Remark VI.4.5]. □

Remarks 3.3. (1) This proof via interpolation domains avoids Alan Loper's difficult argument that the double-boundedness condition is necessary for $\mathrm{Int}(D)$ to be a Prüfer domain [6]. Moreover, it shows that this condition holds in its stronger form.
(2) For a Dedekind domain D with finite residue fields, it is implicit in Alan Loper's characterization that $\mathrm{Int}(D)$ is a Prüfer domain if and only if it is not contained in $D_\mathfrak{m}[X]$ for any maximal ideal $\mathfrak{m}$ of D [6]. This can be related to the fact that, if D is an interpolation domain, then $\mathrm{Int}(D)$ is not contained in $B[X]$ for any overring B of D [Remark 1.8 (1)].

Similarly to the Noetherian case [Corollary 2.6], we finally have the following.

Corollary 3.4. *If the Prüfer domain D is an interpolation domain, then each overring of D is an interpolation domain.*

Proof. If B is an overring of an almost Dedekind domain D, each maximal ideal $\mathfrak{m}$ of B is above a maximal ideal $\mathfrak{n} = \mathfrak{m} \cap D$ of D, and $B_\mathfrak{m} = D_\mathfrak{n}$. If D is an interpolation domain, the double-boundedness condition satisfied by D is then also clearly satisfied by B. □

We end this paper with a question raised by Corollaries 2.6 and 3.4.

Question 3.5. Assume that D is an interpolation domain. Is each overring of D an interpolation domain? In particular, is the integral closure of D an interpolation domain?

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