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INTEGRALLY CLOSED DOMAINS, MINIMAL POLYNOMIALS, AND NULL IDEALS OF MATRICES

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Abstract. We show that every element of the integral closure D' of a domain D occurs as a coefficient of the minimal polynomial of a matrix with entries in D. This answers affirmatively a question of J. Brewer and F. Richman, namely, if integrally closed domains are characterized by the property that the minimal polynomial of every square matrix with entries in D is in $D[x]$. It follows that a domain D is integrally closed if and only if for every matrix A with entries in D the null ideal of A, $N_D(A) = \{f \in D[x] \mid f(A) = 0\}$ is a principal ideal of $D[x]$.

For a square matrix A with entries in a domain D, the null ideal of A in $D[x]$ is $N_D(A) = \{f \in D[x] \mid f(A) = 0\}$. For integrally closed D, W.C. Brown [3] has shown that this ideal is always principal. Conversely, if D is a domain for which all null ideals of matrices are principal, then D is integrally closed. We will show this by demonstrating that every element of the integral closure D' of a domain D occurs as a coefficient of a minimal polynomial of a matrix with entries in D. This also answers a question of Brewer and Richman [2], namely, if integrally closed domains are characterized by the fact that the minimal polynomial of every square matrix with entries in D is in $D[x]$. To put this question in context, we remind the reader of a related, but a priori stronger, property characterizing integrally closed commutative rings:

Fact. *Let R be a commutative ring and T its total ring of quotients. Then R is integrally closed in T if and only if it has the following property:*
whenever f, g, and h are monic polynomials in $T[x]$ with $f(x) = g(x)h(x)$ then $f \in R[x]$ implies $g \in R[x]$ and $h \in R[x]$.

Proof. ($\Leftarrow$) Suppose R has the property and $u \in T$ is integral over R. Let $f \in R[x]$ be a monic polynomial with $f(u) = 0$ then $f(x) = g(x)(x - u)$ for some monic $g \in T[x]$, therefore $(x - u) \in R[x]$ and $u \in R$.

($\Rightarrow$) This is shown in Bourbaki [1, Chpt 5, §1.3, Prop. 11] by means of the splitting ring of g and h. If R is a domain, this direction also follows from the fact that R is an intersection of valuation rings contained in T and in each valuation ring we have Gauß's Lemma $\mathrm{C}(f) = \mathrm{C}(g)\,\mathrm{C}(h)$. □

If D is an integrally closed domain, then the above fact guarantees that the minimal polynomial of every matrix with entries in D is in $D[x]$, since after all the minimal polynomial is a monic factor of the characteristic polynomial.

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Theorem. *Let D be a domain and D' its integral closure. Then every element of D' occurs as a coefficient of a minimal polynomial of a matrix with entries in D.*

Proof. Let K be the quotient field of D and $u \in K$ integral over D. We use the expression "second-highest coefficient" to designate the coefficient of x^{n-1} in a polynomial of degree $n > 0$.

Let $f_1(x)$ be a monic polynomial in $D[x]$ with $f_1(u) = 0$, $\deg f_1 \geq 3$ and second-highest coefficient zero. (Given any monic $f \in D[x]$ with $f(u) = 0$, we can set $f_1(x) = f(x)(x^2 - cx)$, where c is the second-highest coefficient of f.)

We write u as a fraction $u = a/b$ with $a, b \in D$ and set $f_2(x) = f_1(x) + (bx - a)$. Then $f_2(x)$ is another monic polynomial in $D[x]$ with $\deg f_2 \geq 3$, second-highest coefficient zero and $f_2(u) = 0$.

In $K[x]$, $f_1(x) = g(x)(x-u)$ for some monic polynomial $g \in K[x]$ with $\deg g \geq 2$, and $f_2(x) = (g(x)+b)(x-u)$. Note that the second-highest coefficient in both $g(x)$ and $g(x) + b$ is u.

Now let A_i be the companion matrix of f_i for $i = 1, 2$ and A the block-diagonal matrix with A_1 and A_2 on the main diagonal. Then the minimal polynomial $h(x)$ of A is the least common multiple of f_1 and f_2 in $K[x]$. Since $g(x)$ and $g(x)+b$ are relatively prime, the minimal polynomial of A is

$$h(x) = g(x) \big(g(x) + b\big) \big(x - u\big).$$

We have arranged things so that the three monic factors $g(x)$, $g(x)+b$ and $(x-u)$ of $h(x)$ have second-highest coefficients u, u, and $-u$, respectively. Therefore the second-highest coefficient of $h(x)$ is u. □

Corollary. *Let D be a domain. D is integrally closed if and only if the minimal polynomial of every square matrix with entries in D is in $D[x]$.*

If D is a domain with quotient field K and A a square matrix with entries in D, then the following conditions are easily seen to be equivalent (cf. [3]):

i) the minimal polynomial of A, $m_A(x) \in K[x]$, is in $D[x]$.
ii) the null ideal of A in $D[x]$, $N_D(A) = \{f \in D[x] \mid f(A) = 0\}$, is principal.

(To see ($ii \Rightarrow i$): if $N_D(A)$ is principal, it must have a monic generator, since it contains the characteristic polynomial of A, which is monic.) This yields the following variant of our characterization of integral domains:

Variant of Corollary. *Let D be a domain. D is integrally closed if and only if for every square matrix A with entries in D, the null ideal of A in $D[x]$,*

$$N_D(A) = \{f \in D[x] \mid f(A) = 0\}$$

is a principal ideal.

References

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2. J. Brewer and F. Richman, *Weakly integrally closed domains: minimum polynomials of matrices*, Comm. Algebra **28** (2000), 4735–4748.
3. W. C. Brown, *Null Ideals and Spanning Ranks of Matrices*, Comm. Algebra **26** (1998), 2401–2417.

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