

# A GRAPH-THEORETIC CRITERION FOR ABSOLUTE IRREDUCIBILITY OF INTEGER-VALUED POLYNOMIALS WITH SQUARE-FREE DENOMINATOR

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ABSTRACT. An irreducible element of a commutative ring is absolutely irreducible if no power of it has more than one (essentially different) factorization into irreducibles. In the case of the ring  $\text{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$ , of integer-valued polynomials on a principal ideal domain  $D$  with quotient field  $K$ , we give an easy to verify graph-theoretic sufficient condition for an element to be absolutely irreducible and show a partial converse: the condition is necessary and sufficient for polynomials with square-free denominator.

## 1. INTRODUCTION

An intriguing feature of non-unique factorization (of elements of an integral domain into irreducibles) is the existence of non-absolutely irreducible elements, that is, irreducible elements some of whose powers allow several essentially different factorizations into irreducibles [1, 6, 7, 8, 10].

For rings of integers in number fields, their existence actually characterizes non-unique factorization, as Chapman and Krause [4] have shown.

Here, we investigate absolutely and non-absolutely irreducible elements in the context of non-unique factorization into irreducibles in the ring of integer-valued polynomials on  $D$

$$\text{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\},$$

where  $D$  is a principal ideal domain and  $K$  its quotient field.

In an earlier paper [5, Remark 3.9] we already hinted at a graph-theoretic sufficient condition for  $f \in \text{Int}(D)$  to be irreducible. We spell this out more fully in Theorem 1. This condition is not, however, necessary.

We formulate a similar graph-theoretic sufficient condition for  $f \in \text{Int}(D)$  to be absolutely irreducible in Theorem 2, and show a partial converse. Namely, our criterion for absolute irreducibility is necessary and sufficient in the special case of polynomials with square-free denominator, cf. Theorem 3.

First, we recall some terminology. Let  $R$  be a commutative ring with identity.

- (i)  $r \in R$  is called *irreducible* in  $R$  (or, an *atom* of  $R$ ) if it is a non-zero non-unit that is not a product of two non-units of  $R$ .
- (ii) A *factorization* (into irreducibles) of  $r$  in  $R$  is an expression

$$r = a_1 \cdots a_n \tag{1}$$

where  $n \geq 1$  and  $a_i$  is irreducible in  $R$  for  $1 \leq i \leq n$ .

- (iii)  $r, s \in R$  are *associated* in  $R$  if there exists a unit  $u \in R$  such that  $r = us$ . We denote this by  $r \sim s$ .

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(iv) Two factorizations into irreducibles of the same element,

$$r = a_1 \cdots a_n = b_1 \cdots b_m, \quad (2)$$

are called *essentially the same* if  $n = m$  and, after a suitable re-indexing,  $a_j \sim b_j$  for  $1 \leq j \leq m$ . Otherwise, the factorizations in (2) are called *essentially different*.

**Definition 1.1.** Let  $R$  be a commutative ring with identity. An irreducible element  $c \in R$  is called *absolutely irreducible* (or, a *strong atom*), if for all natural numbers  $n$ , every factorization of  $c^n$  is essentially the same as  $c^n = c \cdots c$ .

Note the following fine distinction: an element of  $R$  that is called “*not absolutely irreducible*” might not be irreducible at all, whereas a “*non-absolutely irreducible*” element is assumed to be irreducible, but not absolutely irreducible.

We now concentrate on integer-valued polynomials over a principal ideal domain.

Recall that a polynomial in  $D[x]$ , where  $D$  is a principal ideal domain, is called *primitive* if the greatest common divisor of its coefficients is 1.

**Definition 1.2.** Let  $D$  be a principal ideal domain with quotient field  $K$ , and  $f \in K[x]$  a non-zero polynomial. We write  $f$  as

$$f = \frac{a \prod_{i \in I} g_i}{b},$$

where  $a, b \in D \setminus \{0\}$  with  $\gcd(a, b) = 1$ ,  $I$  a finite (possibly empty) set, and each  $g_i$  primitive and irreducible in  $D[x]$  and call this the *standard form* of  $f$ .

We refer to  $b$  as the *denominator*, to  $a$  as the *constant factor*, and to  $a \prod_{i \in I} g_i$  as the *numerator* of  $f$ , keeping in mind that each of them is well-defined and unique only up to multiplication by units of  $D$ .

**Definition 1.3.** For  $f \in \text{Int}(D)$ , the *fixed divisor* of  $f$ , denoted  $\mathfrak{d}(f)$ , is the ideal of  $D$  generated by  $f(D)$ .

An integer-valued polynomial  $f \in \text{Int}(D)$  with  $\mathfrak{d}(f) = D$  is called *image-primitive*.

When  $D$  is a principal ideal domain, we may, by abuse of notation, write the generator for the ideal, as in  $\mathfrak{d}(f) = c$  meaning  $\mathfrak{d}(f) = cD$ .

**Remark 1.4.** Let  $D$  be a principal ideal domain with quotient field  $K$ , and  $f \in K[x]$  written in standard form as in Definition 1.2. Then  $f$  is in  $\text{Int}(D)$  if and only if  $b$  divides  $\mathfrak{d}(\prod_{i \in I} g_i)$ .

**Remark 1.5.** Let  $D$  be a principal ideal domain with quotient field  $K$ . Then any non-constant irreducible element of  $\text{Int}(D)$  is necessarily image-primitive. Otherwise, if a prime element  $p \in D$  divides  $\mathfrak{d}(f)$ , then

$$f = p \cdot \frac{f}{p}$$

is a non-trivial factorization of  $f$ .

Furthermore,  $f \in K[x] \setminus \{0\}$  (written in standard form as in Definition 1.2) is an image-primitive element of  $\text{Int}(D)$  if and only if (up to multiplication by units)  $a = 1$  and  $b = \mathfrak{d}(\prod_{i \in I} g_i)$ .

**Definition 1.6.** Let  $D$  be a principal ideal domain. For  $f \in \text{Int}(D)$ , and  $p$  a prime element in  $D$ , we let

$$\mathfrak{d}_p(f) = v_p(\mathfrak{d}(f))$$

**Remark 1.7.** By the above definition,

$$\mathfrak{d}(f) = \prod_{p \in \mathbb{P}} p^{\mathfrak{d}_p(f)} \quad \text{and} \quad \mathfrak{d}_p(f) = \min_{c \in D} v_p(f(c))$$

where  $\mathbb{P}$  is a set of representatives of the prime elements of  $D$  up to multiplication by units.

By the nature of the minimum function, the fixed divisor is not multiplicative:

$$d_p(f) + d_p(g) \leq d_p(fg),$$

but the inequality may be strict. Accordingly,

$$d(f)d(g) \mid d(fg),$$

but the division may be strict. Note, however, that

$$d(f^n) = d(f)^n$$

for all  $f \in \text{Int}(D)$  and  $n \in \mathbb{N}$ .

## 2. GRAPH-THEORETIC IRREDUCIBILITY CRITERIA

We refer to, for instance, [2] for the graph theory terms we use in this section.

**Definition 2.1.** Let  $D$  be a principal ideal domain,  $I \neq \emptyset$  a finite set and for  $i \in I$ , let  $g_i \in D[x]$  be non-constant and primitive. Let  $g(x) = \prod_{i \in I} g_i$ , and  $p \in D$  a prime.

- (i) We say that  $g_i$  is *essential* for  $p$  among the  $g_j$  with  $j \in I$  if  $p \mid d(g)$  and there exists a  $w \in D$  such that  $v_p(g_i(w)) > 0$  and  $v_p(g_j(w)) = 0$  for all  $j \in I \setminus \{i\}$ . Such a  $w$  is then called a witness for  $g_i$  being essential for  $p$ .
- (ii) We say that  $g_i$  is *quintessential* for  $p$  among the  $g_j$  with  $j \in I$  if  $p \mid d(g)$  and there exists  $w \in D$  such that  $v_p(g_i(w)) = v_p(d(g))$  and  $v_p(g_j(w)) = 0$  for all  $j \in I \setminus \{i\}$ . Such a  $w$  is called a witness for  $g_i$  being quintessential for  $p$ .

We will omit saying ‘‘among the  $g_j$  with  $j \in I$ ’’ if the indexed set of polynomials is clear from the context.

**Remark 2.2.** When we consider an indexed set of polynomials  $g_i$  with  $i \in I$ , we are not, in general, requiring  $g_i \neq g_j$  for  $i \neq j$ . Note, however, that  $g_i$  being essential (among the  $g_j$  with  $j \in I$ ) for some prime element  $p \in D$  implies  $g_i \not\sim g_j$  in  $D[x]$  for all  $j \in I \setminus \{i\}$ .

**Definition 2.3.** Let  $D$  be a principal ideal domain,  $I \neq \emptyset$  a finite set and for each  $i \in I$ ,  $g_i \in D[x]$  primitive and irreducible.

- (i) The *essential graph* of the indexed set of polynomials  $(g_i \mid i \in I)$  is the simple undirected graph whose set of vertices is  $I$ , and in which  $(i, j)$  is an edge if and only if there exists a prime element  $p$  in  $D$  such that both  $g_i$  and  $g_j$  are essential for  $p$  among the  $g_k$  with  $k \in I$ .
- (ii) The *quintessential graph* of the indexed set of polynomials  $(g_i \mid i \in I)$  is the simple undirected graph whose set of vertices is  $I$ , and in which  $(i, j)$  is an edge if and only if there exists a prime element  $p$  in  $D$  such that both  $g_i$  and  $g_j$  are quintessential for  $p$  among the  $g_k$  with  $k \in I$ .

**Example 2.4.** Let  $I = \{1, 2, 3, 4\}$  and for  $i \in I$ ,  $g_i \in \mathbb{Z}[x]$  as follows:

$$g_1 = x^3 - 19, \quad g_2 = x^2 + 9, \quad g_3 = x^2 + 1, \quad g_4 = x - 5,$$

and set

$$g = (x^3 - 19)(x^2 + 9)(x^2 + 1)(x - 5).$$

A quick check shows that the fixed divisor of  $g$  is 15.

- (i) Taking  $w = 1, 2, 0$  respectively, as witnesses, we see that  $g_2, g_3, g_4$  are quintessential for 5. The polynomial  $g_1$  is not essential for 5 because  $v_5(g_1(a)) > 0$  only if  $a \in 4 + 5\mathbb{Z}$  and for such  $a$ , also  $v_5(g_2(a)) > 0$ .

- (ii) Taking  $w = 1, 0, 2$  respectively, as witnesses, we see that  $g_1, g_2, g_4$  are essential for 3. Only  $g_4$  is quintessential for 3. The polynomial  $g_3$  is not essential for 3.

Figure 1 shows the essential and quintessential graphs of  $(g_1, g_2, g_3, g_4)$ .

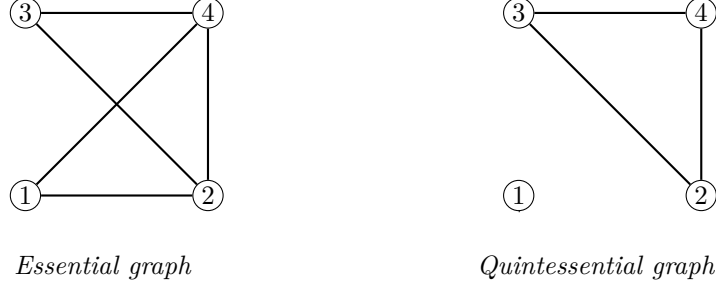


FIGURE 1. Graphs for Example 2.4

**Lemma 2.5.** *Let  $D$  be a principal ideal domain and  $f \in \text{Int}(D)$  a non-constant image-primitive integer-valued polynomial, written in standard form according to Definition 1.2 as*

$$f = \frac{\prod_{i \in I} g_i}{\prod_{p \in T} p^{e_p}},$$

where  $T$  is a finite set of pairwise non-associated primes of  $D$ , and let  $n \in \mathbb{N}$ .

Every  $h \in \text{Int}(D)$  dividing  $f^n$  can be written as

$$h(x) = \frac{\prod_{i \in I} g_i^{\gamma_i(h)}}{\prod_{p \in T} p^{\beta_p(h)}},$$

with  $\gamma_i(h) \in \mathbb{N}_0$  for  $i \in I$  and unique  $\beta_p(h) \in \mathbb{N}_0$  for  $p \in T$ . Moreover, every such representation of  $h$  satisfies:

- (i) If  $q \in T$  and  $j \in I$  such that  $g_j$  is quintessential for  $q$  among the  $i \in I$ , then

$$\beta_q(h) = e_q \gamma_j(h).$$

- (ii) In particular, whenever  $g_j$  and  $g_k$  are both quintessential for the same prime  $q \in T$ , then  $\gamma_j(h) = \gamma_k(h)$ .

*Proof.* We know  $d(f^n) = d(f)^n$  (cf. Remark 1.7). So,  $f^n$  is image-primitive, and, therefore, all polynomials in  $\text{Int}(D)$  dividing  $f^n$  are image-primitive. Let  $f^n = hk$  with  $h, k \in \text{Int}(D)$ . When  $h$  is written in standard form as in Definition 1.2, the fixed divisor of the numerator equals the denominator, and the constant factor is a unit. The same holds for  $k$ . This is so because  $h$  and  $k$  are image-primitive; see Remark 1.5.

Now let  $q \in D$  be prime and  $j \in I$  such that  $g_j$  is quintessential for  $q$ . Note that, by Remark 2.2 and unique factorization in  $K[x]$ , the exponent of  $g_j$  in the numerator of any factor of  $f^n$  is unique.

Writing  $f^n = hk$  as

$$\frac{\prod_{i \in I} g_i^n}{\prod_{p \in T} p^{ne_p}} = \frac{\prod_{i \in I} g_i^{\gamma_i(h)}}{\prod_{p \in T} p^{\beta_p(h)}} \cdot \frac{\prod_{i \in I} g_i^{\gamma_i(k)}}{\prod_{p \in T} p^{\beta_p(k)}},$$

we observe the following equalities and inequalities of the exponents:

- (i)  $ne_q = \beta_q(h) + \beta_q(k)$   
(ii)  $n = \gamma_j(h) + \gamma_j(k)$  and hence  $ne_q = e_q \gamma_j(h) + e_q \gamma_j(k)$

(iii)  $e_q \gamma_j(h) \geq \beta_q(h)$  and  $e_q \gamma_j(k) \geq \beta_q(k)$ .

(i) follows from unique factorization in  $D$ .

(ii) follows from unique factorization in  $K[x]$  and Remark 2.2.

To see (iii), consider a witness  $w$  for  $g_j$  being quintessential for  $q$ . Since  $f$  is image-primitive,  $e_q = v_q(\mathbf{d}(\prod_{i \in I} g_i))$ , by Remark 1.5. From Definition 2.1 and Remark 1.4 we deduce

$$e_q \gamma_j(h) = v_q(g_j(w)) \gamma_j(h) = v_q(g_j^{\gamma_j(h)}(w)) = v_q \left( \prod_{i \in I} g_i(w)^{\gamma_i(h)} \right) \geq \beta_q(h)$$

(and similarly for  $k$  instead of  $h$ ).

Finally, (i) - (iii) together imply  $e_q \gamma_j(h) = \beta_q(h)$  and  $e_q \gamma_j(k) = \beta_q(k)$ .  $\square$

**Theorem 1.** *Let  $D$  be a principal ideal domain with quotient field  $K$ . Let  $f \in \text{Int}(D)$  be a non-constant image-primitive integer-valued polynomial, written in standard form as  $f = g/b$  with  $b \in D \setminus \{0\}$ , and  $g = \prod_{i \in I} g_i$ , where each  $g_i$  is primitive and irreducible in  $D[x]$ .*

*If the essential graph of  $(g_i \mid i \in I)$  is connected, then  $f$  is irreducible in  $\text{Int}(D)$ .*

*Proof.* If  $|I| = 1$ , then  $f$  is irreducible in  $K[x]$ , and, by being image-primitive, also irreducible in  $\text{Int}(D)$ .

Now assume  $|I| > 1$ , and suppose  $f$  can be expressed as a product of  $m$  non-units  $f = f_1 \cdots f_m$  in  $\text{Int}(D)$ . Since  $\mathbf{d}(f) = 1$ , we see immediately that no  $f_i$  is a constant, and that  $\mathbf{d}(f_k) = 1$  for every  $1 \leq k \leq m$ .

Write  $f_k = h_k/b_k$  with  $b_k \in D$  and  $h_k$  primitive in  $D[x]$ . Then  $b = b_1 \cdots b_m$  and there exists a partition of  $I$  into non-empty pairwise disjoint subsets  $I = \bigcup_{i=1}^m I_k$ , such that  $h_k = \prod_{i \in I_k} g_i$ .

Select  $i \in I_1$  and  $j \in I$  with  $j \neq i$ . We show that also  $j \in I_1$ . Let  $i = i_0, i_1, \dots, i_s = j$  be a path from  $i$  to  $j$  in the essential graph of  $(g_i \mid i \in I)$ . For some prime element  $p_0$  in  $D$  dividing  $b$ ,  $g_{i_0}$  and  $g_{i_1}$  are both essential for  $p_0$ . As  $g_i$  is essential for  $p_0$ ,  $p_0$  cannot divide any  $b_k$  with  $k \neq 1$  and, hence,  $p_0$  divides  $b_1$ . For any  $g_k$  essential for  $p_0$  it follows that  $k \in I_1$ , and, in particular,  $i_1 \in I_1$ . The same argument with reference to a prime  $p_k$  for which both  $g_{i_k}$  and  $g_{i_{k+1}}$  are essential, shows for any two adjacent vertices  $i_k$  and  $i_{k+1}$  in the path that they pertain to the same  $I_k$ , and, finally, that  $j \in I_1$ .

As  $j \in I$  was arbitrary,  $I_1 = I$  and  $m = 1$ .  $\square$

**Theorem 2.** *Let  $D$  be a principal ideal domain and  $f \in \text{Int}(D)$  be non-constant and image-primitive, written in standard form as*

$$f = \frac{\prod_{i \in I} g_i}{\prod_{p \in T} p^{e_p}},$$

*where  $I \neq \emptyset$  is a finite set and for  $i \in I$ ,  $g_i \in D[x]$  is primitive and irreducible in  $D[x]$ .*

*If the quintessential graph  $G$  of  $(g_i \mid i \in I)$  is connected, then  $f$  is absolutely irreducible.*

*Proof.* Suppose

$$f^n = \prod_{l=1}^s f_l, \quad \text{where} \quad f_l = \frac{\prod_{i \in I} g_i^{m_l(i)}}{\prod_{p \in T} p^{k_l(p)}}$$

and  $0 \leq m_l(i) \leq n$ ,  $0 \leq k_l(p) \leq ne_p$  and for all  $i$ ,  $\sum_{l=1}^s m_l(i) = n$  and for all  $p$ ,  $\sum_{l=1}^s k_l(p) = ne_p$ .

Fix  $t$  with  $1 \leq t \leq s$ . We show that  $f_t$  is a power of  $f$  by showing that each  $g_i$  with  $i \in I$  occurs in the numerator of  $f_t$  with the same exponent.

Let  $i, j \in I$ . By the connectedness of the quintessential graph, there exists a sequence of indices in  $I$ ,  $i = i_0, i_1, i_2, \dots, i_k = j$  and for each  $h$ , a prime element  $p_h$  in  $T$  such that  $g_{i_h}$  and  $g_{i_{h+1}}$  are both quintessential for  $p_h$ . By Lemma 2.5,  $g_{i_h}$  and  $g_{i_{h+1}}$  occur in the numerator of  $f_t$  with the same exponent. Eventually,  $g_i$  and  $g_j$  occur in the numerator of  $f_t$  with the same exponent, for arbitrary  $i, j \in I$ . In an image-primitive polynomial, the numerator determines its denominator (as in Remark 1.5) and, hence,  $f_t$  is a power of  $f$ . Since  $f_t$  is irreducible,  $f_t = f$ .  $\square$

**Example 2.6.** The binomial polynomial

$$\binom{x}{p} = \frac{x(x-1) \cdots (x-p+1)}{p!}$$

where  $p \in \mathbb{Z}$  is a prime, is absolutely irreducible in  $\text{Int}(\mathbb{Z})$ , by Theorem 2.

The converse of Theorem 2 does not hold in general. For instance, the polynomial

$$f = \frac{x^2(x^2+3)}{4} \in \text{Int}(\mathbb{Z})$$

is absolutely irreducible in  $\text{Int}(\mathbb{Z})$  but the quintessential graph of  $(x, x, x^2+3)$  is not connected.

There is, however, a converse to Theorem 2 in the special case where the denominator of  $f$  is square-free, as we now proceed to show, cf. Theorem 3.

### 3. ABSOLUTELY IRREDUCIBLE POLYNOMIALS WITH SQUARE-FREE DENOMINATOR

Let  $D$  be a principal ideal domain with quotient field  $K$ . When we talk of the denominator of a polynomial in  $K[x]$ , this refers to the standard form of a polynomial introduced in Definition 1.2.

**Remark 3.1.** Let  $D$  be a principal ideal domain. Suppose the denominator of  $f \in \text{Int}(D)$ , written in standard form as in Definition 1.2, is square-free:

$$f = \frac{\prod_{i \in I} g_i}{\prod_{p \in T} p}.$$

Then, if  $f$  is irreducible in  $\text{Int}(D)$ , it follows that each  $g_i$  is essential for some  $p \in T$ . Otherwise, we can split off  $g_i$ . This further implies  $g_i \not\sim g_j$  in  $D[x]$  for  $i \neq j$ , whenever  $f \in \text{Int}(D)$  with square-free denominator is irreducible. A criterion for irreducibility of an integer-valued polynomial with square-free denominator has been given by Peruginelli [9].

**Theorem 3.** *Let  $D$  be a principal ideal domain and  $f \in \text{Int}(D)$  be non-constant and image-primitive, with square-free denominator, written in standard form as*

$$f = \frac{\prod_{i \in I} g_i}{\prod_{p \in T} p},$$

where  $I \neq \emptyset$  is a finite set and for  $i \in I$ ,  $g_i \in D[x]$  is primitive and irreducible in  $D[x]$ .

*Let  $G$  be the quintessential graph of  $(g_i \mid i \in I)$  as in Definition 2.3.*

*Then  $f$  is absolutely irreducible if and only if  $G$  is connected.*

*Proof.* In view of Theorem 2, we only need to show necessity. If  $|I| = 1$ , then  $G$  is connected. Now assume  $|I| > 1$ , and suppose  $G$  is not connected. We show that  $f$  is not absolutely irreducible. If  $f$  is not even irreducible, we are done. So suppose  $f$  is irreducible. This implies  $g_i \not\sim g_j$  in  $D[x]$  for  $i \neq j$ , by Remark 3.1. Since  $G$  is not connected,  $I$  is a disjoint union of  $J_1$  and  $J_2$ , both non-empty, such that there is no edge  $(i, j)$  with  $i \in J_1$  and  $j \in J_2$ .

We express  $T$  as a disjoint union of  $T_1$  and  $T_2$  by assigning every  $p \in T$  for which some  $g_i \in J_1$  is quintessential to  $T_1$ , every  $p \in T$  for which some  $g_i \in J_2$  is quintessential to  $T_2$ , and assigning each  $p \in T$  for which no  $g_i$  is quintessential to  $T_1$  or  $T_2$  arbitrarily. (It may happen that  $T_1 = \emptyset$  and  $T_2 = T$  or vice versa).

Then  $f^3$  factors in  $\text{Int}(D)$  as follows:

$$f^3 = \frac{(\prod_{i \in J_1} g_i)^2 \prod_{j \in J_2} g_j}{(\prod_{p \in T_1} p)^2 \prod_{q \in T_2} q} \cdot \frac{(\prod_{j \in J_2} g_j)^2 \prod_{i \in J_1} g_i}{(\prod_{q \in T_2} q)^2 \prod_{p \in T_1} p}.$$

As  $\text{Int}(D)$  is atomic (cf. [3]), each of the two factors above can further be factored into irreducibles. Since  $J_1$  and  $J_2$  are both non-empty and  $g_i \not\sim g_j$  in  $D[x]$  (and hence,  $g_i \not\sim g_j$  in  $K[x]$ ) for  $i \neq j$ , it is clear that the resulting factorization of  $f^3$  into irreducibles is essentially different from  $f \cdot f \cdot f$ .  $\square$

**Remark 3.2.** Let  $D$  be a principal ideal domain. The proof of Theorem 3 shows that any non-absolutely irreducible element  $f \in \text{Int}(D)$  with square-free denominator exhibits non-unique factorization of  $f^n$  already for  $n = 3$ .

If  $f(x) = \prod_{i \in I} g_i(x)/p$ , where  $D$  is a principal ideal domain,  $p$  a prime of  $D$  and each  $g_i \in D[x]$  primitive and irreducible in  $D[x]$ , then it is easy to see that  $f$  is an irreducible element of  $\text{Int}(D)$  if and only if

- (i)  $d(\prod_{i \in I} g_i(x)) = p$  and
- (ii) each  $g_i$  is essential for  $p$ , that is, for each  $i \in I$  there exists  $w_i \in D$  such that  $v_p(g_i(w_i)) > 0$  and  $v_p(g_j(w_i)) = 0$  for all  $j \in I \setminus \{i\}$ .

An analogous statement relates absolutely irreducible integer-valued polynomials with prime denominator to quintessential irreducible factors of the numerator:

**Corollary 3.3.** Let  $D$  be a principal ideal domain,  $p \in D$  a prime, and  $I \neq \emptyset$  a finite set. For  $i \in I$ , let  $g_i \in D[x]$  be primitive and irreducible in  $D[x]$ . Let

$$f(x) = \frac{\prod_{i \in I} g_i(x)}{p}.$$

Then  $f$  is an absolutely irreducible element of  $\text{Int}(D)$  if and only if

- (i)  $d(\prod_{i \in I} g_i(x)) = p$  and
- (ii) each  $g_i$  is quintessential for  $p$  among the  $g_i$  with  $i \in I$ , that is, for each  $i \in I$  there exists  $w_i \in D$  such that  $v_p(g_i(w_i)) = 1$  and  $v_p(g_j(w_i)) = 0$  for all  $j \in I \setminus \{i\}$ .

*Proof.* If  $d(\prod_{i \in I} g_i(x)) = p$ , then  $f \in \text{Int}(D)$  with  $d(f) = 1$ , and Theorem 3 applies. If, on the other hand,  $f$  is in  $\text{Int}(D)$  and is absolutely irreducible, then  $f$  is, in particular, irreducible and therefore  $d(f) = 1$ , and, again, Theorem 3 applies. Now the statement follows from the fact that, whenever  $d(\prod_{i \in I} g_i(x)) = p$  is prime, the quintessential graph of  $(g_i \mid i \in I)$  is connected if and only if every  $g_i$  is quintessential for  $p$ .  $\square$

We conclude by an example of how to apply Theorem 3:

**Example 3.4.** The following polynomial  $f \in \text{Int}(\mathbb{Z})$  is irreducible, by Theorem 1; but not absolutely irreducible, by Theorem 3:

$$f = \frac{(x^3 - 19)(x^2 + 9)(x^2 + 1)(x - 5)}{15}$$

This is so because the essential graph of  $(x^3 - 19, x^2 + 9, x^2 + 1, x - 5)$  is connected, but the quintessential graph is not connected, see Example 2.4 and Figure 1.

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