13. *A little point-set topology.*

$***$ *Open and closed sets* $***$

13.1 Definition. A subset τ of the power set $\mathcal{P}(X)$ of a set X is called a **topology** on X if the following axioms hold:

(O1) $\emptyset \in \tau$ and $X \in \tau$

(O2) $S, T \in \tau \implies S \cap T \in \tau$

(O3) $S_i \in \tau$ for all $i \in I$ (I an arbitrary index set) $\implies \bigcup_{i \in I} S_i \in \tau$

(X, τ) is then called a **topological space**. The members of τ are called **open sets**.

If (X, τ) is a topological space, then the complements of open sets, $A = X \setminus S$ with $S \in \tau$, are called **closed sets**. The collection of closed sets $\mathcal{C} = \{X \setminus O \mid O \in \tau\}$ satisfies

(C1) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$.

(C2) $A, B \in \mathcal{C} \implies A \cup B \in \mathcal{C}$.

(C3) $A_i \in \mathcal{C}$ for all $i \in I$ (I an arbitrary index set) $\implies \bigcap_{i \in I} A_i \in \mathcal{C}$

Conversely, if $\mathcal{C} \subseteq \mathcal{P}(X)$ satisfys C1–C3, then the complements of elements of $\mathcal{C}$ form a topology on X, whose closed sets are precisely the elements of $\mathcal{C}$.

13.2 Example: Zariski topology on the spectrum of a ring. Let R be a commutative ring. A topology on the spectrum of R ($\mathrm{Spec}(R) = \{P \mid P$ a prime ideal of $R\}$) is defined by specifying its closed sets as sets of prime ideals of the form

$$V(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\},$$

for some ideal I of R.

There's nothing to prevent sets from being open and closed at the same time. Sets both open and closed are often called **clopen**.

13.3 Definition. Let (X, τ) be a topological space.

A collection $\mathcal{B}$ of open sets such that every open set is a union of elements of $\mathcal{B}$ is called a **basis** of the topology τ.

A collection $\mathcal{S}$ of open sets such that every open set is a union of finite intersections of elements of $\mathcal{S}$ is called a **subbasis** of the topology τ.

Topologies can be defined by specifying a basis or subbasis: If $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed with respect to finite intersections, then the unions of (arbitrarily many) members of $\mathcal{B}$ form a topology on X, of which $\mathcal{B}$ is a basis.

If $\mathcal{S} \subseteq \mathcal{P}(X)$ is any collection of subsets of X, then arbitrary unions of finite intersections of members of $\mathcal{S}$ form a topology on X, of which $\mathcal{S}$ is a subbasis.

13.4 Example: Order Topology. If $(X, \leq)$ is a totally ordered set, then **order topology** on X is defined by specifying "open rays", i.e., sets of the form $(a, \infty) = \{x \in X \mid a < x\}$ and $(-\infty, b) = \{x \in X \mid x < b\}$, for $a, b \in X$ as a subbasis.

If $(X, \leq)$ has neither a maximal nor a minimal element, then "open intervals" $(a, b) = \{x \in X \mid a < x < b\}$ form a basis of order topology. (If X does have a maximal or minimal element, then open rays of the form (a, ∞), or $(-\infty, b)$, respectively, have to be added to the open intervals to get a basis.)

13.5 Definition. If (X, τ) is a topological space and Y a subset of X then Y inherits a topological structure from X (called **subspace topology**) through the convention: a subset U of Y is open (in Y) iff there exists an open subset O of X with $U = O \cap Y$.

If Y is an open subset of X then $U \subseteq Y$ is open in Y if and only it is open in X; if Y is a closed subset of X then $A \subseteq Y$ is closed in Y if and only it is closed in X.

13.6 Remark: If $(X, \leq)$ is a totally ordered set and $Y \subseteq X$, then Y inherits a topology from the order topology of X, and at the same time Y inherits an order relation from X which makes $(Y, \leq)$ a totally ordered set, for which order topology may be defined. These two topologies on Y in general do not agree. (Examples can be found among subsets of the real numbers.)

$* * *$ *Neighborhoods and neighborhood bases* $* * *$

Perhaps a more intuitive approach to topology is through neighborhoods of a point, which (as sets containing an open ball around the point) are already familiar from the study of metric spaces.

13.7 Definition. $(*)$ Let (X, τ) be a topological space and $p \in X$. A neighborhood of p is a set U such that there exists an open set O with $p \in O \subseteq U$. The set of all neighborhoods of a point p is called the **neighborhood filter** of p. We will denote it by $\mathcal{U}(p)$.

13.8 Remark: Let (X,τ) be a topological space and $p \in X$. The neighborhood filter of p has the properties

(U1) $\forall U \in \mathcal{U}(p)\ \ p \in U$.

(U2) $U, V \in \mathcal{U}(p) \implies U \cap V \in \mathcal{U}(p)$

(U3) $U \in \mathcal{U}(p)$ and $V \supseteq U \implies V \in \mathcal{U}(p)$

(U4) $\forall U \in \mathcal{U}(p)\ \exists V \in \mathcal{U}(p)\ \forall v \in V\ \ U \in \mathcal{U}(v)$.

Also,

$$(U) \qquad\qquad O \in \tau \iff \forall p \in O\ \ \exists U \in \mathcal{U}(p)\ \ U \subseteq O$$

Conversely, given a set X and for each $p \in X$ a set $\mathcal{U}(p) \subseteq \mathcal{P}(X)$ such that U1–U4 hold, we can define a topolgy τ on X by (U), and, what is more, the neighborhood filter of each point in the resulting topology τ is exactly the $\mathcal{U}(p)$ we started out with.

As with metric spaces, it suffices to know a system of "basic" neighborhoods of a point – with the property that every neighborhood contains one of them – to know all neighborhoods.

13.9 Definition. Let (X,τ) be a topological space. A collection $\mathcal{B}(p) \subseteq \mathcal{U}(p)$ of neighborhoods of p is called a **neighborhood basis** of p if for every $U \in \mathcal{U}(p)$ there exists $B \in \mathcal{B}(p)$ with $B \subseteq U$.

If (X,τ) is a topological space, and for each $p \in X$, $\mathcal{B}(p)$ is a neighborhood basis, then, for every $p \in X$

(B1) $\forall B \in \mathcal{B}(p)\ \ p \in B$.

(B2) $U, V \in \mathcal{B}(p) \implies \exists B \in \mathcal{B}(p)\ \ B \subseteq U \cap V$

(B3) $\forall U \in \mathcal{B}(p)\ \exists V \in \mathcal{B}(p)\ \forall v \in V\ \exists B \in \mathcal{B}(v)\ B \subseteq U$).

Also,

$$(B) \qquad\qquad O \in \tau \iff \forall p \in O\ \ \exists B \in \mathcal{B}(p)\ \ B \subseteq O$$

Conversely, if we are given for every $p \in X$ a collection $\mathcal{B}(p)$ of subsets of X satisfying B1–B3, we can define a topology on X by (B), and in this topology $\mathcal{B}(p)$ will be a neighborhood basis of p.

13.10 Definition. A topological space in which every point has a countable neighborhood basis is said to satisfy the **first countability axiom**.

13.11 Example: Let (X, d) be a metric space, and $B_\varepsilon(p) = \{x \in X \mid d(p, x) < \varepsilon\}$ (for $\varepsilon > 0$, $p \in X$) the open ball of radius ε around p. If we define $\mathcal{B}(p)$ as the set of all $B_\varepsilon(p)$ with $\varepsilon > 0$, then B1–B3 hold. In other words, every metric induces a topology in which the open ε-balls with center p form a neighborhood basis of p. Actually, countably many balls $B_{\frac{1}{n}}(p)$, $n \in \mathbb{N}$, already form a neighborhood basis of p in this topology. We see that every metric space satisfies the first countability axiom.

13.12 Example: I-adic topology Let R be a commutative ring and I an ideal of R. I-adic topology on R is defined by specifying a neighborhood base of $r \in R$:

$$\mathcal{B}(r) = \{r + I^n \mid n \in \mathbb{N}\}.$$

Note that these basic neighborhoods are both open and closed.

$* * *$ *Closure and interior* $* * *$

13.13 Definition. Let A be a subset of a topological space X. The **closure** of A, denoted $\bar{A}$, is defined to be the intersection of all closed sets containing A.

Clearly, $\bar{A}$ is a closed set containing A and every closed set B that contains A also contains $\bar{A}$.

The **closure operator** $A \mapsto \bar{A}$ on $\mathcal{P}(X)$ has the properties:

(A1) $\bar{\emptyset} = \emptyset$.

(A2) $A \subseteq \bar{A}$.

(A3) $\bar{\bar{A}} = \bar{A}$.

(A4) $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

Conversely, any operator $A \mapsto \bar{A}$ on the power set of a set X with properties A1–A4 may be used to define a topology on X by declaring the closed sets to be precisely the sets of the form $\bar{A}$ for some $A \in \mathcal{P}(X)$. (To see that these sets satisfy C3, first note that A4 implies $A \subseteq C \implies \bar{A} \subseteq \bar{C}$. From this we get that $\overline{\bigcap_{i \in I} \bar{A}_i} \subseteq \bar{A}_i$ for all $i \in I$, i.e., $\overline{\bigcap_{i \in I} \bar{A}_i} \subseteq \bigcap_{i \in I} \bar{A}_i$ and the reverse inclusion is just A2.)

13.14 Definition. Let (X, τ) be a topological space and $A \in \mathcal{P}(X)$ then the **open interior** (or **interior** for short) of A, denoted A°, is the union of all open sets contained in A.

Clearly, A° is open and $A^\circ \subseteq A$. Also, $(X \setminus A)^\circ = X \setminus \bar{A}$. (Note that the interior of a non-empty set may be empty.)

∗∗∗ Boundry points, accumulation points and such ∗∗∗

The following properties of the closure of a set (which we defined as the intersection of all closed sets containing it) and the interior of a set (which we defined as the union of all open sets contained in it) are often used as definitions:

13.15 Lemma.
- $\bar{A}$ consists of precisely those points $p \in X$ such that $U \cap A \neq \emptyset$ for every neighborhood $U \in \mathcal{U}(p)$ and
- A° consists of precisely those points $p \in X$ such that there exists a neighborhood $U \in \mathcal{U}(p)$ with $U \subseteq A$.

13.16 Definition. Let (X, τ) be a topological space, $p \in X$ and $A \subseteq X$ then
- p is called a **boundary point** of A if for every neighborhood $U \in \mathcal{U}(p)$ both $A \cap U \neq \emptyset$ and $(X \setminus A) \cap U \neq \emptyset$. The set of all boundary points of A is the **boundary** of A, denoted by δA.
- p is called an **interior point** of A if $p \in A^\circ$, i.e., if there exists a neighborhood $U \in \mathcal{U}(p)$ with $U \subseteq A$.
- p is called an **accumulation point** of A if every neighborhood of p contains an element of A other than p.
- p is called an **isolated point** of A if there exists a neighborhood U of p with $U \cap A = \{p\}$.

Note that interior points of A and isolated points of A are necessarily in A, while boundary points and accumulation points may or may not belong to A.

By purely logical arguments we see that every set A induces a partition of X into three disjoint parts in two different ways ($\dot\cup$ denotes disjoint union):

(i) $X = A^\circ \,\dot\cup\, \delta A \,\dot\cup\, (X \setminus A)^\circ$ and

(ii) $X = \{\text{isolated points of } A\} \dot\cup \{\text{accumulation points of } A\} \dot\cup (X \setminus A)^\circ$.

Also,

(iii) $\bar{A} = A^\circ \,\dot\cup\, \delta A$ and

(iv) $\bar{A} = \{\text{isolated points of } A\} \,\dot\cup\, \{\text{accumulation points of } A\}$

The last two partitions of $\bar{A}$ follow from the characterization of $\bar{A}$ as the set of those p such that $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}(p)$. They are incomparable in the sense that all four combinations of belonging to one set of one partition and one set of the other are possible, an accumulation point of A can be either an interior point or a boundary point of A, etc. Also, we have seen that

(v) $\bar{A} = A \cup \delta A$ and

(vi) $\bar{A} = A \cup \{\text{accumulation points of } A\}$

Unlike (iii) and (iv), the unions (v) and (vi) are in general not disjoint.

$***$ *Continuous functions* $***$

13.17 Definition. Let (X,τ) and (Y,τ') be topological spaces. A function $f\colon X \to Y$ is called **continuous** if $f^{-1}(O)$ is open for every open set $O \subseteq Y$.

13.18 Remark: Inverse image commutes with arbitrary unions and intersections

$$f^{-1}(\bigcup_{i\in I} S_i) = \bigcup_{i\in I} f^{-1}(S_i) \quad \text{and} \quad f^{-1}(\bigcap_{i\in I} S_i) = \bigcap_{i\in I} f^{-1}(S_i).$$

Therefore, for $f\colon X \to Y$ to be continuous, it suffices that $f^{-1}(O)$ be open for all O in some fixed subbasis of Y. Also,

$$f^{-1}(Y \setminus S) = X \setminus f^{-1}(S).$$

Therefore, $f\colon X \to Y$ is continuous if and only if $f^{-1}(A)$ is closed for every closed set $A \subseteq Y$.

In terms of neighborhoods, a function $f\colon X \to Y$ is continuous, if and only if for every $x \in X$, for every $U \in \mathcal{U}(f(x))$ there exists a $V \in \mathcal{U}(x)$ with $f(V) \subseteq U$. The familiar ε-δ-definition of continuous functions is easily seen to be the specialization to metric spaces of this topological characterization.

13.19 Definition. Let (X,τ) and (Y,τ') be topological spaces. A function $f\colon X \to Y$ is called **open** if $f(O)$ is open for every open set $O \subseteq X$.

A bijective function both open and continuous is called a **homeomorphism**.

A topology τ_1 on X is called **stronger** (or **finer**) than another topology τ_2 on the same set X if $\tau_1 \supseteq \tau_2$ (every τ_2-open set is τ_1-open); τ_2 is then called **weaker** or **coarser** than τ_1. Two trivial topologies exist on every set X: **discrete topology** $\tau = \mathcal{P}(X)$ (the finest topology on X) and **chaotic topology** $\tau = \{\emptyset, X\}$ (the coarsest topology on X).

If τ_1 and τ_2 are two topologies on a set X then τ_1 is stronger than τ_2 iff $\mathrm{id}_X : (X,\tau_1) \to (X,\tau_2)$ is continuous; τ_1 is weaker than τ_2 iff $\mathrm{id}_X : (X,\tau_1) \to (X,\tau_2)$ is open.

$***$ *Connectedness* $***$

13.20 Definition. A topological space X is **connected**, if, whenenver O_1 and O_2 are open sets with $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$, it follows that $O_1 = \emptyset$ or $O_2 = \emptyset$. A subset Y of X is connected if it is conneced in subspace topology.

13.21 Exercise. If X is connected and $f\colon X \to Y$ continuous, then $f(X)$ is connected.

13.22 Lemma. If X_i is a connected subset of X for every $i \in I$ (I an arbitrary index set) and $\bigcap_{i\in I} X_i \neq \emptyset$ then $\bigcup_{i\in I} X_i$ is connected.

Proof. Easy exercise. $\square$

13.23 Lemma and Definition. The following relation $\sim$ is an equivalence relation on X: $x \sim y$ if and only if there exists a connected subset C of X with $x, y \in C$.

The equivalence classes with respect to $\sim$ are called the **connected components** of X.

From the above definition it is clear that the connected components of X form a partition of X. Also, the component of $x \in X$ is the union of all connected subsets of X containing x, and it is therefore the unique largest connected subset of X containing x.

13.24 Lemma. If Y is a connected subset of X then every set C with $Y \subseteq C \subseteq \overline{Y}$ is connected.

Proof. Exercise. $\square$

13.25 Definition. A topological space X is **locally connected** if every $x \in X$ has a neighborhood basis consisting of connected neighborhoods.

13.26 Lemma. The connected components of a topological space X are closed sets. If every $x \in X$ has a connected neighborhood (in particular, if X is locally connected) then they are also open.

Proof. By the lemma above, the closure of a connected component is again connected and therefore contained in the component. If a point x possesses a connected open neighborhood U_x then the compoment of x (being the union of all connected sets containing x) contains U_x. $\square$

13.27 Definition. A topological space X is **totally disconnected** if it doesn't contain any connected set with more than one element; equivalently, if its connected components are singletons.

*** *Filters* ***

13.28 Definition. Let X be a set. A **filter** on X is a set $\mathcal{F} \subseteq \mathcal{P}(X)$ with the properties

(1) $\emptyset \notin \mathcal{F}$
(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$ and $X \supseteq C \supseteq A$ then $C \in \mathcal{F}$.

13.29 Definition. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on X. We say that $\mathcal{F}$ is **finer** than $\mathcal{G}$ (or, equivalently, $\mathcal{G}$ is **coarser** than $\mathcal{F}$) if $\mathcal{F} \supseteq \mathcal{G}$.

13.30 Definition. An **ultrafilter** on X is a filter $\mathcal{F}$ with the property:

$$\forall S \subseteq X: \ \ S \in \mathcal{F} \vee (X \setminus S) \in \mathcal{F}.$$

13.31 Definition. If $\mathcal{F}$ is a filter on X then a subset $\mathcal{F}'$ of $\mathcal{F}$ is called a **filter base** of $\mathcal{F}$ if $\forall F \in \mathcal{F} \ \exists F' \in \mathcal{F}'$ with $F' \subseteq F$.

13.32 Remark: If $\mathcal{F}$ is a filter then $\mathcal{F}' \subseteq \mathcal{F}$ is a filter base of $\mathcal{F}$ if and only if $\mathcal{F}$ consists precisely of all supersets of elements of $\mathcal{F}'$. We can easily give a criterion for a collection of sets $\mathcal{F}' \subseteq \mathcal{P}(X)$ to be the base of a filter on X. The set of supersets of elements of $\mathcal{F}'$ is a filter if and only if $\mathcal{F}'$ has the **finite intersection property**, that is, the intersection of any finite number of sets in $\mathcal{F}'$ is non-empty.

13.33 Lemma. If $\mathcal{F}$ is a filter on X containing neither S nor $X \setminus S$, then there exists a filter $\mathcal{F}_1 \supseteq \mathcal{F} \cup \{S\}$ and a filter $\mathcal{F}_2 \supseteq \mathcal{F} \cup \{(X \setminus S)\}$.

Proof. No $F \in \mathcal{F}$ is contained in S (otherwise S would be in $\mathcal{F}$), and likewise, no $F \in \mathcal{F}$ is contained in $X \setminus S$. Therefore $F \cap (X \setminus S) \neq \emptyset$ for all $F \in \mathcal{F}$ and $F \cap S \neq \emptyset$ for all $F \in \mathcal{F}$. Now take $\mathcal{F} \cup \{S\}$ as a filter base for $\mathcal{F}_1$ and $\mathcal{F} \cup \{(X \setminus S)\}$ as a filter base for $\mathcal{F}_2$. $\square$

13.34 Lemma. A filter is maximal with respect to refinement (i.e., inclusion) if and only if it is an ultrafilter.

Proof. It is clear that no sets can be added to an ultrafilter without violating proprty (1) or (2) in the definition of a filter. Conversely, 13.33 shows that a filter that is not an ultrafilter has a proper refinement. $\square$

13.35 Lemma. For every filter $\mathcal{F}$ on X there exists an ultrafilter on X finer than $\mathcal{F}$. $\mathcal{F}$ is the intersection of all ultrafilters on X finer than $\mathcal{F}$.

Proof. Consider the set S of all filters on X containing $\mathcal{F}$, ordered by inclusion. Since the union of a chain of filters is again a filter, every chain in S has an upper bound in S. By Zorn's Lemma, there exists a maximal element in S, which is an ultrafilter containing $\mathcal{F}$, by 13.34. By 13.33, the intersection of all ultrafilters containing $\mathcal{F}$ contains no other sets than the elements of $\mathcal{F}$. $\square$

$*\!*\!*$ *Nets* $*\!*\!*$

13.36 Definition. A **directed set** is a set I With a binary relation $\leq$ such that $\forall i, j, k \in I$

(1) $i \leq i$.

(2) if $i \leq j$ and $j \leq k$ then $i \leq k$.

(3) $\exists n \in I$ with $n \geq i$ and $n \geq j$.

Note that we do not require anti-symmery.

13.37 Definition. A **net** in X is a function from a directed set to X, $\psi\colon I \to X$, usually written (like a sequence) as a list of values indexed by arguments, $(x_i)_{i\in I}$.

13.38 Definition. Let $\psi\colon I \to X$, written as $(x_i)_{i\in I}$, be a net in X, J a directed set and and $\varphi\colon J \to I$ an increasing function that is cofinal in I, that is,

$$\forall i, i' \in I \ \ (i \leq i' \implies \varphi(i) \leq \varphi(i'))$$

$$\forall i \in I \ \ \exists k \in K : \ \ \varphi(k) \geq i.$$

Then the composition of maps $\psi \circ \varphi\colon J \to X$ is called a **subnet** of $(x_i)_{i\in I}$, and is written $(x_{i_j})_{j\in J}$.

13.39 Definition. Let $(x_i)_{i\in I}$ be a net in X and $S \subseteq X$. We say that (x_i) is **eventually in S** if there exists $n \in I$ such that $x_i \in S$ for all $i \geq n$. We say that (x_i) is **frequently in S**, if for all $n \in I$ there exists $i \in I$ with $i \geq n$ and $x_i \in S$.

13.40 Remark: A set of the form $\{x_i \mid i \geq n\}$ for some $n \in I$ is called a **tail** of the net $(x_i)_{i\in I}$. A net is eventually in a set S if and only if some tail is contained in S; it is frequently in S if and only if all its tails intersect S nontrivially.

13.41 Definition. An **ultranet** in X is a net such that for every subset S of X, the net is eventually in S or eventually in $X \setminus S$.

13.42 Exercise. Let $f\colon X \to Y$ be any function. If $(x_i)_{i\in I}$ is an ultranet in X then $(f(x_i))_{i\in I}$ is an ultranet in Y. If $\mathcal{F}$ is an ultrafilter on X then $\{f(F) \mid F \in \mathcal{F}\}$ is an ultrafilter on $f(X)$ and $\{S \subseteq Y \mid \exists F \in \mathcal{F} : \ \ f(F) \subseteq S\}$ is an ultrafilter on Y.

13.43 Definition. Let $\mathcal{F}$ be a filter on X, $x \in X$, and $\mathcal{U}(x)$ the neighborhood filter of x.

$\mathcal{F}$ **converges** to x if and only if $\mathcal{U}(x) \subseteq \mathcal{F}$. In this case, x is called a **limit point** of $\mathcal{F}$.

x is a **cluster point** of $\mathcal{F}$ if and only if $F \cap U \neq \emptyset$ for all $F \in \mathcal{F}$ and all $U \in \mathcal{U}(x)$ (or equivalently, if $x \in \bar{F}$ for all $F \in \mathcal{F}$).

13.44 Proposition. Let $\mathcal{G} \subseteq \mathcal{F}$ be filters on X and $x \in X$.

(1) If $\mathcal{G}$ converges to x then the finer filter $\mathcal{F}$ converges to x.

(2) If x is a cluster point of $\mathcal{F}$ then x is a cluster point of the coarser filter $\mathcal{G}$.

Proof. Follows immediately from the definition of filter convergence and cluster points. □

13.45 Definition. Let $x \in X$ and $\mathcal{U}(x)$ the neighborhood filter of x.

A net in X **converges** to x if and only if for every $U \in \mathcal{U}(x)$, the net is eventually in U. In this case, x is called a **limit point** of the net.

x is a **cluster point** of a net on X if and only if for every $U \in \mathcal{U}(x)$, the net is frequently in U.

13.46 Proposition. Let (x_{n_k}) be a subnet of the net (x_n) on X and $x \in X$. If (x_n) converges to x then the subnet (x_{n_k}) converges to x. If x is a cluster point of the subnet (x_{n_k}) then x is a cluster point of (x_n).

Proof. Follows immediately from the definition of subnet. □

13.47 Proposition. An ultrafilter converges against each of its cluster points. Similarly, an ultranet converges against each of its cluster points.

Proof. Suppose $\mathcal{F}$ is a filter and $x \in X$ such that for all $F \in \mathcal{F}$ and all $U \in \mathcal{U}(x)$, $U \cap F \neq \emptyset$. Then for all $U \in \mathcal{U}(x)$, $(X \setminus U) \notin \mathcal{F}$. If $\mathcal{F}$ is an ultrafilter, $U \in \mathcal{F}$ for all $U \in \mathcal{U}(x)$ follows. The case of nets is similar. □

The following constructions of a net from a filter and a filter from a net often allow to translate statements about filters to statements about nets and vice versa:

13.48 Lemma and Definition. Let $(x_n)_{n\in N}$ be a net on X and $x \in X$. The filter constructed from $(x_n)_{n\in N}$ is defined by taking the set of ends $\{x_n \mid n \geq n_0\}$ for $n_0 \in N$ as a filter basis.

Then the filter constructed from (x_n) converges to x if and only if (x_n) converges to x. Also, x is a cluster point of the filter constructed from (x_n) if and only if x is a cluster point of (x_n).

Proof. Easy exercise. □

13.49 Lemma and Definition. Let $\mathcal{F}$ be a filter on X and $x \in X$. The net constructed from $\mathcal{F}$ is indexed by the set $I = \{(F, y) \mid F \in \mathcal{F}, \quad y \in F\}$ with $(F, y) \geq (F', y') \;:\Longleftrightarrow\; F \subseteq F'$; and $x_{(F,y)} = y$.

Then the net constructed from $\mathcal{F}$ converges to x if and only if $\mathcal{F}$ converges to x. Also, x is a cluster point of the net constructed from $\mathcal{F}$ if and only if x is a cluster point of $\mathcal{F}$.

Proof. Easy exercise. □

13.50 Exercise. Let $\mathcal{F}$ be a filter on X and $x \in X$. For each $F \in \mathcal{F}$ choose $x_F \in F$. Does the net $(x_F)_{F\in\mathcal{F}}$ (indexed by $\mathcal{F}$ directed by $F' \geq F \;:\Longleftrightarrow\; F' \subseteq F$), also satisfy the equivalences of 13.49?

13.51 Remark: For a filter $\mathcal{F}$ to converge to $x \in X$, it suffices that $\mathcal{F}$ contains, for a fixed subbasis S of the topology, every $Y \in S$ with $x \in Y$. Similarly, for a net to converge to x, it suffices that it is eventually in Y for every $Y \in S$ with $x \in Y$.

Proof. Easy exercise. □

$* * *$ *Compactness* $* * *$

13.52 Definition. Let X be a topological space and $Y \subseteq X$. An **open cover** of Y is a set $\mathcal{C}$ of open sets such that $Y \subseteq \bigcup_{O\in\mathcal{C}} O$. Y is **compact** if every open cover of Y admits a finite subcover, that is, there exist $O_1, \ldots, O_n \in \mathcal{C}$ with $Y \subseteq O_1 \cup \ldots \cup O_n$.

Be aware that many authors require compact sets to be Hausdorff, and call our notion of compact "quasi-compact".

13.53 Exercise. If X is compact and $f\colon X \to Y$ continuous, then $f(X)$ is compact.

13.54 Theorem. Let X be a topoological space, and S a subbasis of the topology. The following are equivalent:

(1) X is compact, i.e., every open cover of X has a finite subcover.

(2) Every cover of X consisting of elements of S has a finite subcover.

(3) Every ultrafilter on X converges to some $x \in X$.

(4) Every filter on X has a cluster point $x \in X$.

Proof. $(1 \Rightarrow 2)$ a fortiori.

$(2 \Rightarrow 3)$ Suppose the ultrafilter $\mathcal{U}$ doesn't converge. For every $x \in X$ choose $U_x \in S$ with $x \in U_x$ and $U_x \notin \mathcal{U}$ (possible by 13.51) and let $A_x = X \setminus U_x$. Then $A_x \in \mathcal{U}$. Also, $\{U_x \mid x \in X\}$ covers X so there exists a finite set $Y \subseteq X$ with $\bigcup_{x \in Y} U_x = X$. Therefore $\emptyset = \bigcap_{x \in Y} A_x \in \mathcal{U}$, a contradiction.

$(3 \Rightarrow 4)$ By 13.35, every filter $\mathcal{F}$ on X can be refined to an ultrafilter. This ultrafilter converges to some $x \in X$ and then x is a cluster point of $\mathcal{F}$ by 13.44.

$(4 \Rightarrow 1)$ Suppose $\mathcal{C}$ is an open cover of X that has no finite subcover. Then we may use $\{X \setminus O \mid O \in \mathcal{C}\}$ as base for a filter $\mathcal{F}$ on X. Let $x \in X$ be a cluster point of $\mathcal{F}$ and $O_x \in \mathcal{C}$ with $x \in O_x$. Then $F \cap O_x \neq \emptyset$ for all $F \in \mathcal{F}$. But $(X \setminus O_x) \in \mathcal{F}$, a contradiction. □

13.55 Corollary. Let X be a topologial space. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be a set of closed sets such that every closed subset of X is representable as an arbitrary intersection of finite unions of elements of $\mathcal{B}$ (or, equivalently, such that $\mathcal{S} = \{(X \setminus A) \mid A \in \mathcal{B}\}$ is a subbasis of the topology on X). Then the following are equivalent.

(1) X is compact.

(1') For every set $\mathcal{A} \subseteq \mathcal{P}(X)$ of closed subsets of X it is true that: if $\bigcap_{A \in \mathcal{A}} A = \emptyset$ then there exists a finite subset $\{A_1, \ldots, A_n\} \subseteq \mathcal{A}$ with $\bigcap_{1 \leq i \leq n} A_i = \emptyset$.

(2') Like (1'), but restricted to sets of closed sets $\mathcal{A} \subseteq \mathcal{B}$.

(3') Every ultranet in X converges to some $x \in X$.

(4') Every net in X has a cluster point $x \in X$.

Proof. $(1 \Leftrightarrow 1')$ and $(2 \Leftrightarrow 2')$ by de Morgan. $(3 \Leftrightarrow 3')$ and $(4 \Leftrightarrow 4')$ by 13.48 and 13.49. □

13.56 Exercise. If X is Hausdorff, then we can separate disjoint compact sets C_1, C_2 by open sets, i.e., there are open sets O_1, O_2 such that $C_1 \subseteq O_1$, $C_2 \subseteq O_2$ and $O_1 \cap O_2 = \emptyset$. (First show that we can separate a compact set C from a point $x \in X \setminus C$ by open sets.)

13.57 Exercise.

(i) Every closed subset of a compact space is compact.

(ii) If X is Hausdorff, then every compact subset of X is closed.

$***$ *Product topology* $***$

13.58 Definition. For $i \in I$ (an arbitrary index set) let X_i be a topological space. **Product topology** on the cartesian product $\prod_{i\in I} X_i$ is defined by a subbasis consisting of all sets of the form

$$S(j, O_j) = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_j \in O_j\},$$

for some $j \in I$, and some O_j open $\subseteq X_j$. (Equivalently, the sets O_j could be restriced to members of a given basis or subbasis of the topology of X_j.)

Remark: the finer topology on $\prod_{i\in I} X_i$ given by the basis

$$\mathcal{B} = \{\prod_{i\in I} O_i \mid \forall i\, O_i \text{ open } \subseteq X_i\}$$

is called **box topology**.

Note that the projections $p_j\colon (\prod_{i\in I} X_i) \to X_j$, $p_j((x_i)_{i\in I}) = x_j$, are continuous, both for product topology and for box topology.

13.59 Proposition. A net $(x_\lambda)_{\lambda\in\Lambda}$ in $X = \prod_{i\in I} X_i$ converges to $y = (y_i)_{i\in I}$ in product topology, if and only if for every $i \in I$, its projection to X_i, $(p_i(x_\lambda))_{\lambda\in\Lambda}$ converges to y_i in X_i.

Proof. Easy exercise. □

13.60 Theorem (Tychonoff). $X = \prod_{i\in I} X_i$ (with product topology) is compact if and only if each X_i is compact.

Proof. Easy direction: if $\prod_{i\in I} X_i$ is compact, then for each i, X_i is compact as the image of $\prod_{i\in I} X_i$ under the projection onto the i-th coordinate, which is continuous. Conversely, to show compactness of $X = \prod_{i\in I} X_i$, consider an ultranet on X. The projection to the i-the coordinate is an ultranet on X_i, which converges, since X_i is compact. As all coordinates of the ultranet converge, the ultranet itself converges. □

We give another proof of Tychonoff's theorem using a different criterion for compactness.

Proof. Easy direction: if $\prod_{i\in I} X_i$ is compact and $\mathcal{C}$ is an open cover of X_i, then $\{S(i,O) \mid O \in \mathcal{C}\}$ is an open cover of X, which has a finite subcover $S(i,O_1),\ldots,$ $S(i,O_n)$. Clearly, $O_1,\ldots,$ O_n cover X_i and constitute a finite subcover of $\mathcal{C}$.

Now assuming compactness of each X_i, to show compactness of $X = \prod_{i\in I} X_i$, consider a cover $\mathcal{C}$ by subbasis elements $S(i,O)$. There must be some coordinate $j \in I$ such that the open sets O occurring in sets $S(j,O) \in \mathcal{C}$ cover X_j. (Otherwise, by the axiom of choice, there would be a point $(x_i)_{i\in I}$ such that for all i, x_i is in none of the sets O with $S(i,O) \in \mathcal{C}$, and therefore $(x_i)_{i\in I}$ is not covered by $\mathcal{C}$.) As X_j is compact, there is a finite cover of X_j by open sets $O_1,\ldots O_n$ with $S(j,O_k) \in \mathcal{C}$. Clearly then $S(j,O_1)$, …, $S(j,O_n)$ cover X. □