{ Wh: geg. Sprache erster Ordnung $\mathcal{L}_S$
Eine Struktur für $\mathcal{L}_S$ }

Given a first Order Language $\mathcal{L}_S$
A structure $\mathcal{M}$ for $\mathcal{L}_S$ is $\mathcal{M} = \langle M, \bar{S} \rangle$, where M is a non-empty set and $\bar{S}$ contains for every n-ary function symbol $f \in S$ an n-ary function $\bar{f}: M^n \to M$ and for every n-ary relation symbol $R \in S$ an n-ary relation $\bar{R} \subseteq M^n$.
The assignment of a function to a function symbol and of a relation to a relation symbol that is part of $\mathcal{M}$ is usually denoted by "bar" $\bar{f}$ for corresponding to symbol f.

An interpretation $\mathcal{I}$ is a structure together with an assignment of elements of M to some of the variables of $\mathcal{L}_S$
$i(v_j) = a_j$
$i: \{v_1 \dots v_n\} \to M$
$\mathcal{I} = (\mathcal{M}; v_1 \mapsto a_1, v_2 \mapsto a_2, \dots, v_n \mapsto a_n)$ where $v_1, \dots, v_n$ are pairw. distinct var, $a_1, \dots, a_n \in M$ ($\mathcal{M} = \langle M, \bar{S} \rangle$).

Def) Let $\varphi \in \mathcal{L}_S$ and $\mathcal{I}$ an interpretation of $\mathcal{L}_S$ that assigns el. of M to all var $\in$ free(φ)
Inductively, we define $\mathcal{I} \models \varphi$ "the interpretation $\mathcal{I}$ satisfies φ"

(**Remark:** φ **closed** then a structure $\mathcal{M}$ for $\mathcal{L}_S$ suffice to define $\mathcal{M} \models \varphi$)

First we extend the assignment $v_i \mapsto a_i$ $i = 1, \dots, n$ to a function $\bar{i}: T_{\{v_1, \dots, v_n\}} \to M$ where $T_{\{v_1, \dots, v_n\}}$ is the set of terms of $\mathcal{L}_S$ containing no variables other than $v_1, \dots, v_n$ inductively: given term $t \in T_{\{v_1, \dots, v_n\}}$ if t is a variable then $t = v_j$ for some $j \in \{1, \dots, n\}$ set $\bar{i}(t) = i(v_j) = a_j$
if $t = f x_1, \dots, x_m$ then for each j $x_j = v_{k(j)}$ and
$\bar{i}(t) = \bar{f}(a_{k(1)}, a_{k(2)}, \dots, a_{k(m)}) =$
$= \bar{f}(i(x_1), i(x_2), \dots, i(x_m))$

$i(x_j)$ is defined, because $x_j \in \{v_1, \dots, v_n\}$

Induction via structure of formulas:
if φ is atomic, i.e. $\varphi = R t_1 \dots t_m$
$\mathcal{I} \models \varphi$:iff $(\bar{i}(t_1), \dots, \bar{i}(t_m)) \in \bar{R}$.
if $\varphi = \psi \wedge \theta$ then $\mathcal{I} \models \varphi$:iff $\mathcal{I} \models \psi$ AND $\mathcal{I} \models \theta$.
if $\varphi = (\psi \to \theta)$ then $\mathcal{I} \models \varphi$:iff $\mathcal{I} \models \theta$ OR $\mathcal{I} \not\models \psi$.
$\mathcal{I} \models \theta$ OR (NOT $\mathcal{I} \models \psi$)
etc.

if $\varphi = \forall x \psi$ then $\mathcal{I} \models \varphi$:iff FOR EVERY $a \in M$ $\mathcal{I} / x \mapsto a \models \psi$
where $\mathcal{I} = (\mathcal{M}, v_1 \mapsto a_1, v_2 \mapsto a_2, \dots, v_n \mapsto a_n)$
and if $x \notin \{v_1, \dots, v_n\}$ then $\mathcal{I} / x \mapsto a$ is $(\mathcal{M}, v_1 \mapsto a_1, \dots, v_n \mapsto a_n, \underline{x \mapsto a})$
and if $x = v_j$ then $\mathcal{I} / x \mapsto a$ is $(\mathcal{M}, v_1 \mapsto a_1, \dots, v_{j-1} \mapsto a_{j-1}, v_j \mapsto a, v_{j+1} \mapsto a_{j+1}, \dots, v_n \mapsto a_n)$

If $\varphi = \exists x \psi$ then
$\mathcal{I} \models \varphi$:iff there exists $a \in M$ s.t. $\mathcal{I} / x \mapsto a \models \psi$ ($\mathcal{I} / x \mapsto a$ as above)

Def) A closed formula $\varphi \in \mathcal{L}_S$ is called **universally valid** iff every structure $\mathcal{M}$ of $\mathcal{L}_S$ satisfies φ. Let φ, ψ closed formulas ψ is a (semantic) consequence of φ iff for every structure $\mathcal{M}$ with $\mathcal{M} \models \varphi$ it is true that $\mathcal{M} \models \psi$.
We write $\varphi \models \psi$. (No danger of confusion with $\mathcal{M} \models \psi$ because we can tell if a structure or a formula is before the symbol $\models$).
Closed formulas φ, ψ are logically equivalent (written as $\varphi \dashv\models \psi$ or $\varphi \sim \psi$): iff $\varphi \models \psi$ and $\psi \models \varphi$
(in other words iff every structure $\mathcal{M}$ satiesfies φ if and only if it satisfies ψ).

Remark: If φ, ψ are closed formulas $\in \mathcal{L}_S$ then
$\varphi \models \psi$ iff $\varphi \to \psi$ universally valid
<u>left:</u> $\forall \mathcal{M} \, (\mathcal{M} \models \varphi \Rightarrow \mathcal{M} \models \psi)$
<u>right:</u> $\forall \mathcal{M} \quad \mathcal{M} \models \varphi \to \psi$

Similarly: $\varphi \dashv\models \psi$ iff $\forall \mathcal{M} \quad \mathcal{M} \models \varphi \leftrightarrow \psi$
$\mathcal{M}$ structure for $\mathcal{L}_S$
both sides are true iff for every structure φ, ψ are both true or both false.

For formulas $\in \mathcal{L}_S$ possibly containing free variables:
If free $(\varphi) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_n}\}$ $i_1 < i_2 < \ldots < i_n$ then the universal closure of φ is the formula
$$\forall v_n \forall v_{n-1} \ldots \forall v_2 \forall v_1 \; \varphi.$$

A formula φ is called **universally valid** iff its universal closure is universally valid.
The equivalent of the remark above does not hold for formulas that are not closed:
take $x>0$ and $y>0$
If we define $\mathcal{M} \models \varphi$ for by $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models$ universal closure of φ then $\varphi \dashv\models \psi$ does not imply for all $\mathcal{M}$: $\mathcal{M} \models \varphi \leftrightarrow \psi$ example
$\varphi = x>0$ $\psi = y>0$ then for all $\mathcal{M}$: $\mathcal{M} \models \forall x \; x>0$ iff $\mathcal{M} \models \forall y \; y>0$ but not for all $\mathcal{M} \models (x>0) \to (y>0)$ because that would mean $\mathcal{M} \models \forall x \forall y \, (x>0 \to y>0)$

EX: prove or disprove: for formulas possibly containing free var:
If we define "φ universally valid" by:
$\forall \mathcal{J} \; \mathcal{J} \models \varphi$ (where $\mathcal{J}$ runs through all interpretations assigning el. of M to all free var. of φ) and we define $\varphi \models \psi$ by for all $\mathcal{J}$ with $\mathcal{J} \models \varphi$ it is true that $\mathcal{J} \models \psi$ (where $\mathcal{J}$ runs through all interpretations that assign el. of M to all var $\in$ free$(\varphi) \cup$ free(ψ) and $\varphi \dashv\models \psi$ iff $\varphi \models \psi$ AND $\psi \models \varphi$
then the above remarks w. interp. instead of structure are TRUE.

Def) A theory $T \subseteq \mathcal{L}_S$ is a set of closed formulas:
A model of $T \subseteq \mathcal{L}_S$ is a structure $\mathcal{M}$ for $\mathcal{L}_S$ with $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

<u>Technical definitions:</u> substructures, homomorphisms, ...

Given a structure $\mathcal{M} = \langle M, \bar{S} \rangle$ for $\mathcal{L}_S$ and a non-empty subset $N \subseteq M$ then $\mathcal{N} = \langle N, \bar{S}|_N \rangle$ is a <u>substructure</u> of M iff for every n-ary relation symbol R of $\mathcal{L}$ $R^N = \bar{R} \cap N^n$
(where $\bar{R}$ is the relation in $\bar{S}$ corresponding to the symbol n) and for every f n-ary function symbol $\in \mathcal{L}_S$ N is closed under $\bar{f}$ (if $a_1, \dots, a_n \in N$ then $\bar{f}(a_1, \dots, a_n) \in N$) and $\bar{f}^n = \bar{f}|_N$ (the restriction of $\bar{f}: M^n \to M$ to $\bar{f}^N = \bar{f}|_N : N^n \to N$), and $\bar{S}|_N$ assigns $\bar{f}|_N$ to the function symbol f and $\bar{R} \cap N^n$ to the relation symbol R for all symbols in S.

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Def) for formulas (possibly containing free variables):
F, G *universally equivalent*: iff $F \leftrightarrow G$ is universally valid.

Def) for formulas possibly containing free var. F *universally valid*: iff the universal closure of F
($\forall v_{i_n} \forall v_{i_{n-1}} \dots \forall v_{i_1}\ F$, where v_{i_n} are the var. occurring free in F $i_1 < i_2 < \dots < i_n$)

<u>Example</u>) "$x = x$ is true" means $\forall x\ x = x$ is true.

Remark: for closed formulas F, G universally equivalent:
previous def. by $\forall$ structure $\mathcal{M}$
$\mathcal{M} \models F$ iff $\mathcal{M} \models G$ is equivalent to later def.
$\forall \mathcal{M}$ $\mathcal{M} \models F \leftrightarrow G$.

Def) a theory is a set of <u>closed</u> formulas

Def) if T is a theory and F a formula, F is called a *logical consequence* of T iff every structure $\mathcal{M}$ satisfying T satisfies the universal closure of F.
$\forall M$ with $\forall G \in T$ $\mathcal{M} \models T$ it is the case that $\mathcal{M} \models$ universal closure of F.

Formal proofs / derivations

Given a first order language L and a theory T in L (T set of closed formulas $\subseteq L$). The set of theorems in T, $Thm_L(T)$ is the subset of L inductively defined as follows: 1) certain logical axioms are in $Thm_L(T)$;

2) $T \subseteq Thm_L(T)$

3) $Thm_L(T)$ is closed under certain rules.

Our choice of logical axioms

I) tautologies: a formula resulting from a tautology of propositioned logic (containing the var $A_1, \dots, A_n$) by replacing A_i by F_i,
F_i an arbitrary formula of our language L, is called a topology L.

$A_1 \to (A_2 \vee \neg A_2)$ $\qquad F_1 = \forall x \; x > y \qquad F_2 = z \equiv w$

$\forall x : x > y \to (z \equiv w \; \vee \neg(z \equiv w))$

Easy to show all tautologies are universally valid.

II) Quantification axioms

1) for arbitrary $F \in \mathcal{L}$ and an arbitrary variable the following is an axiom:

$$\exists x \, F \leftrightarrow \neg(\forall x (\neg F))$$

Easy to see by Def of $\mathcal{M} \models \exists x \, F$ and $\mathcal{M} \models \forall x \, F$ these axioms are universally valid.

2) for arbitrary $F, G \in \mathcal{L}$ and x a variable <u>not occuring free</u> in F the following is an axiom

$$\forall x \, (F \to G) \to (F \to \forall x \, G)$$

EX: find examples with x free in F such that the resulting formula is not universally valid.

Remark: if x not free in F such formulas can be shown to be univ. valid by routine induction on structure.

II) Let F be a formula t a term, x a variable such that
3) the following is an axiom: $\forall x\, F \rightarrow F[\frac{t}{x}]$
no free occurence of x in F is in the scope of a quantifier.

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Def) [scope of a quantifier]:
First notice the every occurence of a quantifier $Q \in \{\forall, \exists\}$ in a formula F is the first symbol of a unique subformula of F whose second symbol is a variable. This var. is the var. bound by the quantifier. The scope of the quantifier is: all free occurences of x in F as well as the occurence of x immetiatly following the quantifier.

Q_2 scope of Q_2

Example $\exists x((\forall x\ x>y) \vee x=z)$

scope of Q_1

Q_1

(EX) find examples of F and t such that $\forall x\, F \rightarrow F[\frac{t}{x}]$ is not universally valid.

If x not free in F inside the scope of a quantifier binding a var. in t then $\forall x\, F \rightarrow F[\frac{t}{x}]$ can be shown to be universally valid with the help of the following lemma:

(*)
Lemma: Let $\mathcal{M}$ be a structure for L, F a formula, t a term, x a variable of L
t containing no variables except from $u_1, \dots, u_n$
F containing no free var other than possibly $u_1, \dots, u_n, w_1, \dots, w_m$
If x does not occur free in F in the scope of a quantifier binding any u_i then
The following are equivalent

$\mathcal{M};\ u_1 \mapsto a_1, \dots, u_n \mapsto a_n,\ w_1 \mapsto b_1, \dots, w_m \mapsto b_m \quad \models F[\frac{t}{x}]$

$\mathcal{M};\ u_1 \mapsto a_1, \dots, u_n \mapsto a_n,\ w_1 \mapsto b_1, \dots, w_m \mapsto b_m,\ x \mapsto \overline{t} \models F$
where $\overline{t} = \overline{i}(t)$ is the interpretation of t in
$\mathcal{M};\ u_1 \mapsto a_1, \dots, u_n \mapsto a_n$

Rules:

(A) modus ponens
if F and $F \to G$ in Thm then also G in Thm

Remark: if $\mathcal{M} \models F$ and $\mathcal{M} \models F \to G$ then $\mathcal{M} \models G$

(B) generalization: for F arbitrary $\in \mathcal{L}$, x a variable
if $F \in$ Thm then $\forall x\ F$ in Thm.

if for formulas possibly containing free var. we define
$\mathcal{M} \models F$ by $\mathcal{M} \models$ universal closure of F
and we observe that for all structures $\mathcal{M}$ and F formula
not containing free occ. of x $\mathcal{M} \models F$ equivalent to
$\mathcal{M} \models \forall x\ F$ then we see that this rule is valid, i.e.

if $\mathcal{M} \models F$ then $\mathcal{M} \models \forall x\ F$.

Def) If t is a theory in L then $\underline{\mathrm{Thm}_L(T)}$ is the intersection of all subsets of L containing the logical axioms and T and closed under the rules (A) & (B).

Notation: if T is a theory of L and $F \in \mathcal{L}$ we write
$T \vdash_L F$ for $F \in \mathrm{Thm}_L(T)$.

Lemma: $T \vdash_L F$ if and only if there exists a finite sequence of formulas $F_1, \dots, F_n$ in L such that $F_n = F$ and for each i, either $F_i \in T \cup \{$ Logical axioms $\}$ or F_i is derivable from F_j, F_k for some $j, k < i$ by rule A [i.e. $F_j = F_k \to F_i$ or $F_k = F_j \to F_i$]
or F_i derivable from some F_j with $j < i$ by rule B [i.e. $F_i = \forall x\ F_j$ for some var. x].

Prf: EX