Starting with distr. complementary lattice $(L, \wedge, \vee)$ define

$$r \cdot s = r \wedge s \quad \text{and}$$
$$r + s = (r \wedge s^c) \vee (s \wedge r^c)$$

(alternative, equivalent def:)
$$r + s = (r \vee s) \wedge (r \wedge s)^c$$

EX in compl. distr. lattice
$$(r \wedge s^c) \vee (s \wedge r^c) = (r \vee s) \wedge (r \wedge s)^c$$

Show $(L, +, \cdot)$ is Boolean algebra

idempotence: $r \cdot r = r$ because $r \wedge r = r$, lattice axiom

assoc. of $\cdot$ is assoc. of $\wedge$ (lattice ax.)

(*) neutral element of + is 0. (smallest el. of L)

commutativity of + follows from comm. of $\vee$

(*) to show $r + 0 = r$

$r + 0 = (r \wedge 1) \vee (0 \wedge r^c) = r \vee 0 = r$ ✓

neutral element of $\cdot$ is 1 (max. el. of L)

$r \cdot 1 = r \wedge 1 = r$ ✓

inverse with respect to (w. r. t.) + : $-r = r$

$r + r = (r \wedge r^c) \vee (r \wedge r^c) = 0 \vee 0 = 0$

to check associativity of +:

$$r + (s + t) = r + ((s \wedge t^c) \vee (t \wedge s^c)) =$$
$$= (r \wedge [(s \wedge t^c) \vee (t \wedge s^c)]^c) \vee ([s \wedge t^c) \vee (t \wedge s^c)] \wedge r^c)$$

transform using lattice axioms and de Morgan until you get a formula symmetric in r and t then we have shown
$r + (s + t) = t + (s + r)$
commutativity of + implies
$r + (s + t) = (r + s) + t$

distr. $r(s + t) = r \cdot s + r \cdot t$
check using distr. of $\wedge$ over $\vee$ and vice versa and de Morgan.

EX Construction of Boolean algebra of Boolean alg. from distr. compl. lattice and construction of distr. complementary lattice from Boolean alg. are inverse to each other.

omitted definition: S a set of connectives of prop. logic $S \subseteq \{\wedge, \vee, \rightarrow, \leftrightarrow, |, \neg\}$ then $\mathcal{L}_S$ sub-language of $\mathcal{L}$ consisting of formulas containing no other connectives but those in S.

$\mathcal{L}_S$ complete : $\Leftrightarrow$ $\forall F \in \mathcal{L}$ $\exists F' \in \mathcal{L}_S$: $F \Leftrightarrow F'$.

Def) A sub-algebra of a Boolean algebra $(R, +, \cdot)$ is a subring (containing 1_R)

Example $Y \subseteq X$ sets, $\mathcal{P}(Y) \subseteq \mathcal{P}(X)$ but $\mathcal{P}(Y)$ is not a sub-algebra of the Boolean alg. $\mathcal{P}(X)$, because $1_{\mathcal{P}(X)} = X \notin \mathcal{P}(Y)$.

Remark: if S set $\subseteq R$ Boolean alg. For S to be sub-alg. it suffices to check : $\forall s, t \in S$ $s+t$, $s \cdot t \in S$ and $1_R \in S$.
in general, for $S \subseteq R$, R ring with 1 to check that S subring containing 1 it suffices to check $\forall s, t \in S$: $s-t$, $s \cdot t$, $1_R \in S$.

We have shown : S set $\subseteq R$ Boolean alg. S is sub alg. iff $1_R \in S$ and $s, t \in S \Rightarrow s \wedge t$, $s \vee t \in S$.
because S closed under $\wedge, \vee$ implies closed under $+, \cdot$ $(r \cdot s = r \wedge s$, $r + s = (r \wedge s^c) \vee (s \wedge r^c))$ and S closed under $+, \cdot$ implies S closed under $\wedge, \vee$ complement by $(r \wedge s = r \cdot s$, $r \vee s = r + s + rs$, $r^c = (1 + r))$

We will show: Every boolean algebra is isomorphic to a sub-algebra of some "set algebra" $(\mathcal{P}(S), \cap, \cup)$.

EX : show that there are Boolean algebras not isomorphic to any set algebra $(\mathcal{P}(S), \cap, \cup)$
(hint: set-alg. always has atoms)

Def)
$y \in (L, \wedge, \vee)$ is an atom, if $y \neq 0$ and $\nexists z \in L$ $0 < z < y$
$(a < b : \Leftrightarrow a \leq b \wedge a \neq b)$.

Def) A Boolean alg. then $(\mathcal{P}(A), \cap, \cup)$ is Boolean alg.
As a ring $(\mathcal{P}(A), +, \cdot)$ is $\prod_{a \in A} \{0, 1\}$ (every $Y \subseteq A$ identified with characteristic function

$$\chi_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases} \qquad \chi_Y : X \mapsto \{0, 1\}$$

with $+, \cdot$ (corresponding to symmetric difference and in section of sets $\chi_{A \Delta B} = \chi_A + \chi_B$
$\chi_{A \cap B} = \chi_A \cdot \chi_B$ addition, multiplication of functions coordinatewise.)

Let $S(A)$ Stone space of the boolean alg. A be the subset of homomorphismus in $\prod_{a\in A}\{0,1\}$

$\mathcal{P}(A) = \prod_{a\in A}\{0,1\}$ is the set of all functions $f: A \to \{0,1\}$ f written as $(f(a))_{a\in A}$, since we have $+, \cdot$ an A (Boolean alg.) and an $\{0,1\}$ $(+, \cdot$ mod 2, field operations on $\mathbb{F}_2 \simeq (\mathbb{Z}/2\mathbb{Z}, +, \cdot)$

$S(A) = \{ f \in \prod_{a\in A}\{0,1\} \mid f: A \to \{0,1\}$ ringhomomorphismus$\}$

i.e. $f(a+b) = f(a) + f(b)$
$f(a\cdot b) = f(a)\cdot f(b)$
$f(1) = 1$

EX $S(A)$ is not a sub-algebra of $\prod_{a\in A}\{0,1\}$

Theorem: The subset of $S(A)$ consisting of all open-closed subsets of A in $S(A)$ [consisting of all char-functions of open-closed sets in $S(A)$] is a Boolean alg. isomorphic to A.

To show this theorem and also the compactness. Thus of prop. logic:
Let $\mathcal{L}$ be the language of prop logic and $F \subseteq \mathcal{L}$ a set of formulas.
Then F is satisfiable ($\exists$ assignment of truth values to the truth variables that makes all formulas in F true) iff every <u>finite</u> sub-set of F is satisfiable.
(This holds also if we have more than countably infinitely many truth var. – for any cardinality of the set of truth variables.)]

Topology

Given set S: a topology on S is a collection $\mathcal{T}$ of subsets of S called the "open sets" satisfying the following axioms;

(O1) $\emptyset, S \in \mathcal{T}$

(O2) $O_1, O_2 \in \mathcal{T} \Rightarrow O_1 \cap O_2 \in \mathcal{T}$

(O3) for arbitrary index set I

$$\bigcup_{i\in I} O_i \in \mathcal{T} \quad \text{provided} \quad O_i \in \mathcal{T} \text{ for all } i \in I.$$

Given S with topology $\mathcal{T}$ a set $A \subseteq S$ is called "closed" iff $S \setminus A$ is open.

One could also define a topology by specifying the collection of closed sets $\mathcal{A}$ provided $\mathcal{A}$ satisfies:
(A1) $\emptyset, S \in \mathcal{A}$
(A2) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
(A3) for arbitrary index set I
$\Rightarrow (\forall i\ A_i \in \mathcal{A}) \Rightarrow \bigcap_{i\in I} A_i \in \mathcal{A}$

Def) (S, τ) topological space, $s \in S$.
Then T set $\subseteq S$ is a neighborhood of s iff $\exists O \in \tau$ s.t. $s \in O \subseteq T$.

Denote by $\mathcal{U}(s)$ the set of all neighborhoods of s in (S, τ). Then the following hold:

(N1) $\forall U \in \mathcal{U}(s) \quad s \in U$
(N2) $U_1, U_2 \in \mathcal{U}(s) \Rightarrow U_1 \cap U_2 \in \mathcal{U}(s)$
(N3) $U \in \mathcal{U}(s)$, T set $\subseteq S$ then $U \subseteq T \Rightarrow T \in \mathcal{U}(s)$
(N4) $\forall U \in \mathcal{U}(s) \quad \exists V$ with $s \in V \subseteq U$ s.t. $\forall v \in V \quad U \in \mathcal{U}(v)$.

One could also specify a topology by giving, for every $s \in S$, a set $\mathcal{U}(s) \subseteq \mathcal{P}(s)$ s.t. N1 - N4 hold.

The **open sets** then are defined as sets O s.t. $\forall s \in O \quad \exists U \in \mathcal{U}(s) \quad U \subseteq O$.

These constructions of open sets through neighborhoods and of neighborhoods through open sets are inverse to each other.

Def) (S, τ) top. space. $\mathcal{B} \subseteq \tau$.
$\mathcal{B}$ is a "base" or "basis" of the topology τ iff $\tau = \{ \bigcup_{B \in \mathcal{e}} B \mid \mathcal{e} \subseteq \mathcal{B} \}$ (The open sets are exactly the unions of sets in $\mathcal{B}$.)

Let $\mathcal{S} \subseteq \tau$. $\mathcal{S}$ is a **sub-basis** of τ iff the finite intersections of sets in $\mathcal{S}$ form a basis of τ.
(The open sets are exactly the (arbitrary) unions of finite intersections of sets in $\mathcal{S}$.)

For example: a basis of metric space $\mathbb{R}^n$ (Euklidean metric) are $B_{\frac{1}{r}}(a)$ $r \in \mathbb{N}$, $a \in \mathbb{R}^n$ [countable basis]

subbasis: sets of the form

$$\{(a_1, \dots, a_n) \mid a_i > q\} = U_{i,q} \qquad p \in \mathbb{Q}$$
$$1 \leq i \leq n$$
$$\{(a_1, \dots, a_n) \mid a_i < q\} = L_{i,q} \qquad q \in \mathbb{Q}$$

(S,τ) topological space, a neighborhood base of $s \in S$ is a set of neighborhoods of s $\mathcal{B}(s) \subseteq \mathcal{U}(s)$ s.t. $\forall U \in \mathcal{U}(s)$ $\exists B \in \mathcal{B}(s)$ $B \subseteq U$.

A neighborhood base of s satisfies:

(N1) $\forall B \in \mathcal{B}(s)$ $s \in B$
(N2) $A, B \in \mathcal{B}(s)$ $\exists C \in \mathcal{B}(s)$ $C \subseteq A \cap B$
(N3) $\forall B \in \mathcal{B}(s)$ $\exists C \in \mathcal{B}(s)$ $C \subseteq B$ s.t. $\forall c \in C$ $\exists D \in \mathcal{B}(s)$ $D \subseteq B$.

Conversely, given, for every $s \in S$ a collection of sets $\mathcal{B}(s)$ satisfying N1 – N3 we can define a topology on S by defining $O \subseteq S$ is open iff $\forall s \in O$ $\exists B \in \mathcal{B}(s)$ $B \subseteq O$.

Example in a metric space "ε-balls" are basic neighborhoods $B_\varepsilon(s) = \{t \in S \mid d(s,t) < \varepsilon\}$ $\mathcal{B}(s) = \{B_\varepsilon(s) \mid \varepsilon > 0\}$

for example I-adic topology on a ring R (where I is an ideal of R)
For $r \in R$ $\mathcal{B}(r) = \{ r + I^n \mid n \in \mathbb{N}\}$.

Product topology

Given topological spaces S_i, $i \in I$ (arbitrary index set)
We define a topology on $\prod_{i \in I} S_i$ by specifying a subbasis $\mathcal{S} = \{ S_{j,O} \mid j \in I,\ O \text{ open} \subseteq S_j \}$ where
$S_{j,O} = \{ (s_i)_{i\in I} \mid s_j \in O\}$

Remark: a basis consists of sets $B_{i_1,\dots,i_n,O_1,\dots,O_n} = S_{i_1 O_1} \cap \dots \cap S_{i_n O_n} =$

$= \{ (s_i)_{i \in I} \mid s_{i_j} \in O_j \text{ for } 1 \le j \le n\}$

This topology is called product topology on $\prod_{i\in I} S_i$.

For comparison: box topology on $\prod_{i \in I} S_i$ defined by basis of consisting of prod. of open sets for all i
O_i open $\subseteq S_i$ $\prod_{i\in I} O_i = \{ (s_i) \mid \forall i\ s_i \in O_i\}$

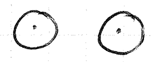
Box topology is finer than product top.
A topology τ is finer than a top. τ' (on the same set S) iff $\tau' \subseteq \tau$. "every τ' open set is τ-open".

Compactness

(S,τ) is called **compact**, iff, for all open coverings of S there exists a finite subcovering, i.e. whenever $S = \bigcup_{i \in I} O_i$, O_i open $\subseteq S$, there exists $i_1, \dots, i_n$ (finitely many) s.t. $S = O_{i_1} \cup \dots \cup O_{i_n}$.

By de Morgan equivalent:
A_i closed $\subseteq S$ for $i \in I$ s.t. $\bigcap_{i \in I} A_i = \emptyset$ then there exist $i_1, \dots, i_n$ (finitely many) s.t. $\bigcap_{k=1}^{n} A_{i_k} = \emptyset$.

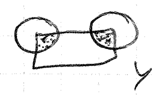
Def) (S,τ) is called **Hausdorff** (T_2) iff $\forall s,t \in S$ $s \neq t$ $\exists U \in \mathcal{U}(s), V \in \mathcal{U}(t)$ with $U \cap V = \emptyset$



Be aware that many authors call our concept of compactness "quasi-compact" and demand Hausdorff in addition to the covering property in their def. of compact.

Subspace topology

Given (X,τ) and $Y \subseteq X$, we define a topology on Y by defining $Z \subseteq Y$ is open (in subspace top., on Y) iff $\exists Z'$ open $\subseteq X$ with $Z = Z' \cap Y$.



So we call a subset Y of X compact, iff Y with subspace top. is a compact space, equivalently Y compact $\subseteq X$ iff whenever $Y \subseteq \bigcup_{i \in I} O_i$ O_i open $\subseteq X$ for all i there exist $i_1, \dots, i_n$ $Y \subseteq O_{i_1} \cup \dots \cup O_{i_n}$.

<u>Tychonoff's theorem</u> (w/o proof)
Let S_i be a compact space for each $i \in I$ (arbitrary index set) then $\prod_{i \in I} S_i$ is compact ($\prod_{i \in I} S_i$ <u>carries product top.</u>, doesn't work for box top.

Actually $\prod_{i \in I} S_i$ with prod. top is compact iff $\forall i$ S_i is compact.

(Ex) If S is Hausdorff and C_1, C_2 compact $\subseteq S$, $C_1 \cap C_2 = \emptyset$ then $\exists O_1, O_2$ open with $C_1 \subseteq O_1$, $C_2 \subseteq O_2$, $O_1 \cap O_2 = \emptyset$.
(First consider special case $C_2 = \{c\}$.)

(Ex) Use Tychonoff's theorem to show compactness th of propositional logic
If $F \subseteq \mathcal{L}$ is not satisfyable then $\exists F'$ finite $\subseteq F$ s.t. F' not satisfyable. This also holds for definitions of $\mathcal{L}$ with more than countably many truth variables works for $\{A_i \mid i \in I\}$ I arbitrary.

Stone's theorem

A Boolean algebra, then consider $\prod_{a \in A} \{0,1\} = \{f : A \to \{0,1\} \mid f \text{ function}\}$.
This is a Boolean algebra with addition, multi coordinate wise mod 2.

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x) \cdot g(x)$$

add, mult on $\{0,1\}$ are add./mult. mod 2.

By identifying $f : A \to \{0,1\}$ with the set $\{a \in A \mid f(a) = 1\}$ (subset of A of which f is the characteristic function) we can identify $\prod_{a \in A} \{0,1\} = \{0,1\}^A$ with $(\mathcal{P}(A), \cap, \cup)$.

Define the **stone space** $S(A) := \{f \in \{0,1\}^A \mid f : A \to \{0,1\} \text{ homom.}\}$ where homomorphism means ring homom. $A \to \mathbb{Z}/2\mathbb{Z}$ with $f(1) = 1$. equiv. Bool. alg. homom.

$S(A)$ is a **topological space** with subspace topology inherited from product topology on $\prod_{a \in A} \{0,1\}$.

The **canonical basis** of the topology on $S(A)$ consists of sets of the form for $a_1, \dots, a_n \in A$, and $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$
$B = \{h \in \operatorname{Hom}(A, \{0,1\}) \mid f(a_i) = \varepsilon_i \quad 1 \le i \le n\}$

Def) S set, ~~des~~ **discrete topology** on S is $\tau = \mathcal{P}(S)$.
(Singletons $\{s\}$ form a basis of τ), Whenever we talk about an unspecified topology on a finite set, we mean discrete top.

Ex F_i finite set for all $i \in I$.
Then a basis of product top. on $\prod_{i \in I} F_i$ is given by sets of the form
$\{(x_i)_{i \in I} \mid \text{for } j = 1, \dots, n \quad x_{i_j} = f_{i_j}\}$
for some $\{i_1, \dots, i_n\}$ finite $\subseteq I$ and some $f_i \in F_i$.
And these basis open sets are also **closed**.

Canonical basic open sets of $S(A)$ are open and closed "**clopen**".

Lemma: A Boolean alg. $S(A)$ Stone space. $B = \{f \in \operatorname{Hom}(A, \{0,1\}) \mid f(a_i) = \varepsilon_i \text{ for } i = 1 \dots n\}$
(for $a_1, \dots, a_n \in A$, $\varepsilon_1, \dots, \varepsilon_n \in \{0,1\}$)
Then there exists a unique $a \in A$ s.t.
$B = \{f \in \operatorname{Hom}(A, \{0,1\}) \mid f(a) = 1\}$

Prf: <u>existence:</u> $f \in \text{Hom}(A, \{0,1\})$ then $f(a)=0$ iff $f(1+a)=1$.

So there exist $b_1, \dots, b_n \in A$ s.t.
$B = \{ f \in \text{Hom}(A, \{0,1\}) \mid f(b_i) = 1 \quad i = 1, \dots, n\}$
namely for $\varepsilon_i = 1$ take $b_i = a_i$,
for $\varepsilon_i = 0$ take $b_i = 1 + a_i$

Now $f(b_1) = 1 \wedge \dots \wedge f(b_n) = 1$ iff
$f(b_1) \cdot f(b_2) \dots \cdot f(b_n) = 1$ $\qquad b = b_1 \dots b_n$
iff $f(b_1 \cdot \dots \cdot b_n) = 1$. So $B = \{ f \in \text{Hom} \mid f(b) = 1\}$

<u>Uniqueness:</u> $f \in \text{Hom}(A, \underset{\{0,1\}}{\overset{\shortparallel}{\mathbb{F}_2}})$. Since f as a

Boolean alg. hom. satisfies $f(0)=0$, $f(1)=1$

f is surjective. Since $\mathbb{F}_2$ is a field, $\ker f$ is a max. ideal of A, $\ker f = \{ a \in A \mid f(a) = 0\}$.

We have a bijective correspondence
$\{\text{max-ideals of } A\} \leftrightarrow \{\text{homomorphisms} : A \to \mathbb{F}_2\}$
given by $\ker f \longleftarrow f$
$M \longmapsto \pi_M : A \to \{0,1\} = A/M$

To show uniqueness of a in $B = \{ f \mid f(a) = 1\}$ ~~$= \{ f \in \text{Hom}(A, \mathbb{F}_2) \mid f(b) = 1\}$~~. we show
for $a \neq b$ $\quad \{ f \in \text{Hom}(A, \mathbb{F}_2) \mid f(a) = 1 \} \neq$
$\{ f \in \text{Hom}(A, \mathbb{F}_2) \mid f(b) = 1\}$

We have to show $\exists$ hom of f with $f(a) = 0$, $f(b) = 1$
or $f(a) = 1$, $f(b) = 0$

equivalent to $f(a+b) = 1$
equivalent to $f(a+b+1) = 0$
since $a+b+1 \neq 1$ $\quad (a \neq b \Rightarrow a+b \neq 0)$, $a+b+1$ is not a unit in A, the principal ideal $(a+b+1)$

[(EX) the only unit (invertible element) in a Boolean alg. is 1.]

is not A. In a ring A with 1, every ideal $I \neq A$ is contained in a maxi. ideal M.

Take M a max ideal of A with $a+b+1 \in M$ and f the unique hom: $A \to \{0,1\}$ with $\ker f = M$. Then $f(a+b+1) = 0$, $f(a+b) = 1$, $f(a) \neq f(b)$.

Remark: A Bool. alg. $\varphi : A \to \{\text{canonical basic open sets of } S(A)\}$

$\varphi(a) = \{ f \in \text{Hom}(A, \{0,1\}) \mid f(a) = 1\}$
is bijective by this Lemma.