10. *Primary ideals and the radical.*

∗∗∗ Multiplicatively closed setes. ∗∗∗

10.1 Definition. A non-empty subset S of a commutative ring R is called **multiplicatively closed**, or **multiplicative**, if s, $t \in S$ implies $st \in S$.

A multiplicatively closed set S is called **saturated** if it contains every divisor of each of its elements, i.e., if $s = ab \in S$ implies $a \in S$.

The set $\bar{S} = \{t \in R \mid \exists s \in S \;\, t \bigm| s\}$ of divisors of elements of a multiplicatve set S – clearly the smallest saturated multiplcative set containing S – is called the **saturation** of S.

Examples of multiplicatively closed sets (1–3 are saturated):

1) $R \setminus P$, where P is a prime ideal of R,
2) $R \setminus \bigcup_{\alpha \in A} P_\alpha$, where all P_α are prime ideals of R
3) the set of all non-zerodivisors in R
4) the set of powers of a fixed element $\{r^n \mid n \in \mathbb{N}_0\}$.

10.2 Lemma. Let S be a multiplicatively closed subset of R and I an ideal of R with $I \cap S = \emptyset$. Then there exists a prime ideal P of R with $I \subseteq P$ and $P \cap S = \emptyset$.

Proof. Apply Zorn's Lemma to $\mathcal{J} = \{J \trianglelefteq R \mid I \subseteq J \text{ and } S \cap J = \emptyset\}$. Since $I \in \mathcal{J}$, $\mathcal{J} \neq \emptyset$. Every chain of ideals in $\mathcal{J}$ has an upper bound in $\mathcal{J}$: its union is in $\mathcal{J}$. Let P be a maximal element of $\mathcal{J}$. We will show that P is prime. $S \neq \emptyset$ implies $P \neq R$. Suppose $ab \in P$, $a \notin P$ and $b \notin P$. By maximailty of P there exists $s \in S \cap (P + Ra)$ and $t \in S \cap (P + Rb)$. But then $st \in S \cap P$, a contradiction. □

10.3 Corollary.

(i) Let S be a saturated multiplicatively closed subset of R. Then $S = R \setminus \bigcup_{P \in \mathcal{P}} P$, where $\mathcal{P}$ is the set of all prime ideals P of R with $S \cap P = \emptyset$.

(ii) The saturated multiplicative subsets of R are precisely the complements of unions of prime ideals. (The trivial case $S = R$ corresponds to the empty union of prime ideals).

Proof. Ad (i). Clearly, $S \subseteq R \setminus \bigcup_{P \in \mathcal{P}} P$. For the reverse inclusion, we show that for every $u \notin S$ there exists some prime ideal P with $S \cap P = \emptyset$ and $u \in P$. S being saturated, $u \notin S$ implies $Ru \cap S = \emptyset$. By Lemma 10.2 there exists a prime ideal P with $Ru \subseteq P$ and $S \cap P = \emptyset$. Since the complement of a union of prime ideals is a saturated multiplicateve set, (ii) follows. □

10.4 Corollary. The set $\mathcal{Z}(R)$ of zero-divisors of a commutative ring R is a union of prime ideals of R. More generally, the set $\mathcal{Z}(M)$ of zero-divisors of any R-module M is a union of prime ideals of R.

Proof. Verify that $R \setminus \mathcal{Z}(R)$, and more generally, $R \setminus \mathcal{Z}(M)$, is a saturated multplicative set. $\square$

10.5 Lemma. Let I be an ideal of R and P a prime ideal containing I. Then there exists a minimal prime P' of I with $I \subseteq P' \subseteq P$.

Proof. Let $S = R \setminus P$ and consider the set $\mathcal{S}$ of multiplicatively closed sets T with $S \subseteq T$ and $T \cap I = \emptyset$. Then $\mathcal{S}$ is non-empty because $S \in \mathcal{S}$. Every chain of elements of $\mathcal{S}$ has an upper bound in $\mathcal{S}$ (its union). Let S_0 be a maximal element of $\mathcal{S}$ (which exists by Zorn's Lemma), and set $U = R \setminus S_0$. By Lemma 10.2, there exists a prime ideal P' containing I and disjoint from S_0, i.e., $I \subseteq P' \subseteq U$. It follows that $P' = U$ and P' is a minimal prime of I, since for any prime ideal Q with $I \subseteq Q \subsetneqq U$, its complement $R \setminus Q$ would be an element of $\mathcal{S}$ stricltly containing S_0. $\square$

$* * *$ *The radical.* $* * *$

10.6 Definition. Let I be an ideal of R. The **radical of** I is defined by

$$\sqrt{I} = \{r \in R \mid \exists n \in \mathbb{N} \ \ r^n \in I\}.$$

Convince yourself of the following easy facts:

10.7 Remark:
(1) $I \subseteq \sqrt{I}$
(2) $\sqrt{\sqrt{I}} = \sqrt{I}$
(3) $\sqrt{I} = R \iff I = R$
(4) If Q is an intersection of prime ideals then $\sqrt{Q} = Q$.

Recall that an element $r \in R$ is called **nilpotent** if there exists a natural number n with $r^n = 0$.

10.8 Definition. The radical of (0), consisting of all nilpotent elements of R, is called the **nilradical** of R:

$$\mathrm{Nil}(R) = \{r \in R \mid \exists n \in \mathbb{N} \ \ r^n = 0\}.$$

10.9 Proposition.

$$\sqrt{I} = \bigcap_{\substack{P \text{ prime} \\ I \subseteq P}} P.$$

Proof. Clearly, $\sqrt{I}$ is contained in all prime ideals containing I. Conversely, if $r \notin \sqrt{I}$, then $S = \{r^n \mid n \in \mathbb{N}\}$ is a multiplicatively closed set with $S \cap I = \emptyset$ and by Lemma 10.2, there exists a prime ideal P with $I \subseteq P$ and $r \notin P$. □

10.10 Corollary. The nilradical of R is the intersection of all prime ideals of R.

10.11 Lemma. Some rules of arithmetic for radicals:

(1) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$

(2) $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$

(3) $\sqrt{I} + \sqrt{J} = R \iff I + J = R$

Proof. Exercise. □

10.12 Lemma. Let P be a prime ideal of R and I an ideal of R.

(i) $P^n \subseteq I \subseteq P$ for some $n \in \mathbb{N}$ implies $\sqrt{I} = P$.

(ii) If R is Noetherian, then $\sqrt{I} = P$ implies for some n $P^n \subseteq I \subseteq P$.

Proof. (i) is easy. Ad (ii), $P = (p_1, \ldots, p_m) = \sqrt{I}$ implies for some $k \in \mathbb{N}$, $p_1^k, \ldots, p_m^k \in I$. Every element q of P is an R-linear combination of the p_i. By the multinomial theorem, $(r_1 p_1 + \ldots + r_m p_m)^m k$ is a sum of terms each of which is divisible by p^k for some i, therefore $q^n \in I$ for every $q \in P$, $n = km$. □

$* * *$ *Primary ideals* $* * *$

10.13 Definition. An ideal Q of R is called **primary** iff $Q \neq R$ and for all $a, b \in R$ with $ab \in Q$ either $a \in Q$ or $\exists n \in \mathbb{N}$ $b^n \in Q$.

The definition of primary tends to be confusing at first. If Q is primary, then $ab \in Q$ implies $a \in Q$ or $b \in \sqrt{Q}$, as well as, by symmetry, $b \in Q$ or $a \in \sqrt{Q}$. Therefore, for an ideal $Q \neq R$, being primary is equivalent to

$$ab \in Q \Rightarrow a \in Q \vee b \in Q \vee a, b \in \sqrt{Q}$$

10.14 Easy exercise. If Q is a primary ideal then $\sqrt{Q}$ is a prime ideal.

10.15 Definition. A primary ideal Q with $\sqrt{Q} = P$ is called **P-primary**.

Not every ideal whose radical is prime is primary, however. For instance, $\sqrt{P^n} = P$ for every prime ideal P and $n \in \mathbb{N}$, but not every power of a prime ideal is primary. In a more positive vein, we have the following facts:

10.16 Lemma. Let M be a maximal ideal of R and Q an ideal with $\sqrt{Q} = M$. Then Q is M-primary.

Proof. Suppose $ab \in Q$ and $a \notin M$; we must show $b \in Q$. We have $M + aR = R$, i.e., $\sqrt{Q} + aR = R$. This implies $Q + aR = R$, so $1 = q + ar$ for some $q \in Q$ and $r \in R$, and finally $b = qb + abr \in Q$. $\square$

10.17 Corollary. Let M be a maximal ideal of R and $I \neq R$ an ideal with $M^n \subseteq I$ for some $n \in \mathbb{N}$. Then I is M-primary.

Proof. I is not contained in any prime ideal P other than M: if $I \subseteq P$ then for every $m \in M$, $m^n \in I \subseteq P$ implies $m \in P$, hence $M = P$. Since $I \neq R$, this shows $\sqrt{I} = M$ and we are done. $\square$

Just like prime ideals, primary ideals behave nicely with respect to ring homomorphisms:

10.18 Lemma. Let $f\colon R \to T$ be a ring homomorphism (with $f(1) = 1$, as always) and I, Q ideals of T. Then

(1) $f^{-1}(\sqrt{I}) = \sqrt{f^{-1}(I)}$.

(2) If Q is P-primary then $f^{-1}(Q)$ is $f^{-1}(P)$-primary.

Again, like prime ideals, primary ideals can be characterized by a property of their quotient ring.

10.19 Proposition. An ideal Q is primary if and only if R/Q is a non-zero ring in which every zero-divisor is nilpotent.

Proof. Easy. $\square$

Together with the previous lemma this implies that a canonical projection induces a bijection between primary ideals containing the kernel and primary ideals in the quotient ring. (Note that everything we show for canonical projections holds mutatis mutandis for any surjective ring homomorphism $f\colon R \to T$, since f is just the canonical projection $\pi\colon R \to R/\mathrm{Ker} f$ followed by the isomorphism $\bar{f}\colon R/\mathrm{Ker} f \to T$.)

10.20 Proposition. Let I be an ideal of R and $\pi\colon R \to R/I$ the canonical projection. Then a bijection between all primary ideals Q of R with $I \subseteq Q$ and all primary ideals of R/I is given by $Q \mapsto \pi(Q)$. Its inverse is $\tilde{Q} \mapsto \pi^{-1}(\tilde{Q})$ (for $\tilde{Q} \trianglelefteq R/I$).

Primary ideals also behave just like prime ideals with respect to the canonical homomorphism into a ring of fractions $f_S\colon R \to R_S$ and with respect to the canonical projection onto a residue class ring:

10.21 Proposition. Let R_S be a ring of fractions of R and $f_S\colon R \to R_S$ the canonical homomorphism.

(1) If $Q \trianglelefteq R$ is primary and $\frac{r}{s} \in Q_S$ then $r \in Q$.

(2) A bijection between the primary ideals Q of R with $S \cap Q = \emptyset$ and all primary ideals of R_S is given by $Q \mapsto Q_S$ (for Q a primary ideal of R). Its inverse is $\tilde{Q} \mapsto f_S^{-1}\tilde{Q}$ (for $\tilde{Q}$ a primary ideal of R_S).

This is shown just like the analogous statement for prime ideals. (Note that $Q \cap S = \emptyset$ implies $\sqrt{Q} \cap S = \emptyset$.)