7. *Noetherian Rings.*

7.1 Proposition. Let R be a commutative ring. Then the following are equivalent:

(1) Every ideal of R is finitely generated.

(2) Every ascending chain of ideals of R is of finite length.

(3) Every non-empty set of ideals of R has a maximal element.

Proof. $(1 \Rightarrow 2)$ and $(2 \Rightarrow 3)$ are easy.
$(3 \Rightarrow 1)$ Let I be an ideal of R and $\mathcal{S}$ the set of finitely generated ideals contained in I. $\mathcal{S}$ (being non-empty because it contains (0)) has a maximal element J. Suppose $J \neq I$ and pick $i \in I \setminus J$. Then $iR + J \subseteq I$ is finitely generated and strictly contains J, a contradiction. $\square$

7.2 Definition. A commutative ring satisfying one and hence all of the equivalent conditions of the preceeding theorem is called Noetherian.

Condition (2) above is called "the ascending chain condition" (ACC) (for ideals). Similar conditions for objects other than ideals, or for special kinds of ideals, often play a rôle. For instance, a generalization of Noetherian domains is given by requiring the ascending chain condition only for divisorial ideals; the resulting domains are called Mori domains.

7.3 Definition. Let R be a commutative ring, M an R-module. M is called **Noetherian** if every R submodule of M is finitely generated.

Since the ideals of a ring R are precisely the R-submodules of R, a ring R is Noetherian (as a ring) if and only if it is a Noetherian R-module.

7.4 Remark: As in Proposition 7.1, the following are equivalent:

(1) M is a Noetherian R-module

(2) Every ascending chain of R-submodules of M is of finite length.

(3) Every non-empty set of R-submodules of M contains a maximal element.

7.5 Lemma. Let M be am R-module and L a submodule of M. Then M is Noetherian if and only if both L and M/L are Noetherian.

Proof. If M is Noetherian, then L is Noetherian, since every submodule of L is s submodule of M. Also, M/L is Noetherian, since every submodule of M/L is of the form $\pi(N)$, where N is a submodule of M and $\pi\colon M \to M/L$ the canonical projection, and $\pi(N)$ is generated by the images under π of a set of generators of N.

Conversely, suppose L and M/L are Noetherian. Given a submodule N of M, let a $n_1, \dots n_s$ be generators of $N \cap L$ and $m_1 + L, \dots, m_t + L$ generators of $(N + L)/L$ with $m_1, \dots, m_t \in N$. Then N is generated by $n_1, \dots n_s, m_1, \dots, m_t$. □

7.6 Proposition. For every Noetherian ring R, every finitely generated R-module is Noetherian.

Proof. We only need to show that every free R-module of finite rank is Noetherian, since every R-module generated by n elements is a homomorphic image of the free R-module of rank n.

Induction on the rank of M: For $n = 1$, $M = R$, and R is Noetherian. For $\operatorname{rank}(M) = n > 1$, $M = R \oplus L$, L a free R-module of rank $n - 1$. L is Noetherian by induction hypothesis, $M/L = R$ is Noetherian, and therefore M is Noetherian by the previous Lemma. □

7.7 Theorem (Hilbert's basis theorem). Let R be a Noetherian ring. Then $R[x]$ is Noetherian.

Proof. Let $J \neq (0)$ be an ideal of $R[x]$. For $n \in \mathbb{N}$, let I_n be the set of leading coefficients of polynomials of degree n or less in J and set $I = \bigcup_{n \in \mathbb{N}} I_n$. The I_n are ideals of R with $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ and therfore I is an ideal of R too. The fact that both I and all I_n are finitely generated now implies that J is finitely generated: let $g_1, \dots, g_m \in J$ be polynomials whose leading coefficients generate I and let $N = \max_{1 \leq k \leq m} \deg g_k$. To $g_1, \dots, g_m$ we add for each $n < N$ a finite set of elements of J of degree n or less whose leading coefficients generate I_n. That the resulting finite set of polynomials generates J is now easily seen by induction on the degree of an element of J. □

7.8 Corollary. Let R be a Noetherian ring. Then $R[x_1, \dots, x_n]$ and therefore every finitely generated R-algebra (as a homomorphic image of a polynomial ring $R[x_1, \dots, x_n]$) is Noetherian.

7.9 Lemma. Let R be a commutative ring and P an ideal of R that is not finitely generated and which is maximal with respect to this property. Then P is prime.

Proof. Suppose $ab \in P$ and $a \notin P$, $b \notin P$. By maximality of P, $P + (a)$ is finitely generated, by elements $p_1 + r_1 a$, …, $p_k + r_k a$ (with $p_i \in P$, $r_i \in R$), and similarly,

$P+(b)$ is finitely generated, by elements $q_1+s_1b, \ldots, q_m+s_mb$ (with $q_i \in P$, $s_i \in R$).

Also, the ideal $(P:a)=\{r \in R \mid ra \in P\}$ properly containes P (because $P \subset P+(b) \subseteq (P:a)$), and is therefore finitely generated, by $j_1, \ldots, j_h$, say.

If $c \in P$, then $c \in P+(a)$ and there exist $t_1, \ldots, t_k \in R$ with $c = t_1p_1 + \ldots + t_kp_k + (t_1r_1 + \ldots + t_kr_k)a$. Let $d = t_1r_1 + \ldots + t_kr_k$, then $da = c - (t_1p_1 + \ldots + t_kp_k) \in P$ and therefore $d \in (P:a)$. Let $t_1', \ldots, t_h' \in R$ such that $d = t_1'j_1 + \ldots + t_h'j_h$.

Then $c = t_1p_1 + \ldots + t_kp_k + t_1'j_1a + \ldots + t_h'j_ha$, and $p_1, \ldots, p_k$ as well as $j_1a, \ldots, j_ha$ are in P. We have shown that $p_1, \ldots, p_k, j_1a, \ldots, j_ha$ generate P, a contradiction. □

7.10 Theorem (Cohen). Let R be a commutative ring. If every prime ideal of R is finitley generated, then R is Noetherian.

Proof. The set of non-finitely-generated ideals of a commutative ring clearly satisfies the hypothesis of Zorn's lemma. Therefore, if this set is non-empty, it has a maximal element by Zorn's lemma. Such a maximal element is a prime ideal by the previous lemma. Therefore, every non-Noetherian ring has a prime ideal that is not finitely generated. □