5. *Rings of Quotients and Localization*

A **multiplicatively closed** subset of a ring is a set $S \neq \emptyset$ such that $st \in S$ whenever $s, t \in S$. A multiplicatively closed set S is **saturated**, if S contains all divisors of every $s \in S$. The saturation of a multiplicatively closed set S is $\bar{S} = \{t \in R \mid \exists r \in R \ \ rt \in S\}$.

5.1 Definition. If S is a multiplicatively closed subset of a commutative ring R and $0 \notin S$ then the **ring of fractions of R with denominators in** S, $(R_S, +, \cdot)$ and the canonical homomorphism $f_S \colon R \to R_S$ are defined as follows:

R_S is the set of equivalence classes on $S \times R$ under the equivalence relation

$$(s, r) \sim (s', r') \ :\Longleftrightarrow \ \exists t \in S \ \ tsr' = ts'r.$$

The equivalence class of (s, r) is denoted by $\frac{r}{s}$. Addition and multiplication are defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{s'r + sr'}{ss'} \qquad \text{and} \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

and the canonical homomorphism $f_S \colon R \to R_S$ by $f_S(r) = \frac{rs}{s}$ (for any $s \in S$).

A lot of facts have to be checked before the above definition can be called valid, for instance that $\sim$ is really an equivalence relation, that addition and multiplication are well-defined (i.e. the resulting equivalence class depends only on the equivalence classes of the arguments and not on the numerator and denominator of the fractions representing them), that R_S satisfies the ring axioms, and that f_S is well-defined and a ring homomorphism. These verifications are easy, if tedious. In doing them, one inevitably encounters the following properties of R_S:

5.2 Remark:

(1) We do not exclude any interesting cases by stipulating $0 \notin S$. If $0 \in S$ then the construction would yield $R_S = \{0\}$.
(2) For all $r \in R$ and $s, t \in S$ $\quad \frac{r}{s} = \frac{rt}{st}$.
(3) R_S is a commutative ring with zero $\frac{0}{s}$ and identity $\frac{s}{s}$ (for any $s \in S$).
(4) $\frac{r}{s} = \frac{r'}{s'}$ iff there exists a $t \in S$ with $trs' = tr's$.
(5) If S contains no zero-divisors, then $\frac{r}{s} = \frac{r'}{s'}$ iff $rs' = r's$.
(6) $\frac{r}{s} = 0$ iff there exists a $t \in S$ with $rt = 0$.
(7) If S contains no zero-divisors, then $\frac{r}{s} = 0$ iff $r = 0$.
(8) $\frac{r}{s}$ is a unit iff there exists a $t \in S$ with $rt \in S$.
(9) If S is saturated, then $\frac{r}{s}$ is a unit iff $r \in S$.

(10) For every $s \in S \quad f_S(s)$ is a unit in R_S.
(11) $\mathrm{Ker} f_S = \{r \in R \mid \exists t \in S\, rt = 0\}$.
(12) f_S is injective iff S contains no zero-divisor.

5.3 Lemma. Let S be a multiplicatively closed subset of R and $\bar{S}$ its saturation. Then $\varphi\colon R_S \to \bar{S}^{-1}R$ given by $\varphi(\frac{r}{s}) = \frac{r}{s}$ is an isomorphism of rings.

Proof. φ is well-defined and a ringhomomorphism; to see injectivity, suppose $\frac{r}{s} \in \mathrm{Ker}\varphi$, then $rt = 0$ for some $t \in \bar{S}$. t divides some $t' \in S$, such that $t'r = 0$ and therefore $\frac{r}{s} = 0$ in R_S. For surjectivity, consider $\frac{r}{t}$ with $r \in R$ and $t \in \bar{S}$. For some $s \in S$ and $w \in R$ we have $s = tw$ (and thus $w \in \bar{S}$) and $\frac{r}{t} = \frac{rw}{tw} = \frac{rw}{s} = \varphi(\frac{rw}{s})$. □

5.4 Proposition. All rings of quotients of R are of the form form R_S with $S = R \setminus \bigcup_{P \in \mathcal{P}} P$ for a set $\mathcal{P}$ of prime ideals of R.

Proof. By the previous lemma, every ring of quotients is of the form R_S with S saturated. By Lemma 10.3, every saturated multiplicative set is the complement of a union of prime ideals and vice versa. □

The importance of the construction lies in the relationship between ideals of R and ideals of R_S. We already know that for every ideal I of R, $f_S(I)R_S$ is an ideal of R_S - the extension of I into R_S. We denote this ideal by I_S.

5.5 Definition. Names and notation for examples of rings of fractions:

- If S is the set of all non-zero-divisors of R, then R_S is called the **total ring of fractions of** R; a special case of this is the **quotient field** of an integral domain, which is R_S with $S = R \setminus \{0\}$.
- If $S = R \setminus P$ for a prime ideal P of R, then R_S is denoted R_P and called the **localization of** R **at** P. The ideal I_S of R_P is denoted I_P or IR_P.

5.6 Remark: The apparent ambiguity of notation in the preceding definition – R_S the ring of fractions with denominators in S, and R_P the ring of fractions with denominators in the complement $R \setminus P$ – we resolve by the following convention: if $0 \notin S$ then R_S denotes the ring of fractions of R with denominators in S, but if $0 \in S$, e.g., if S is a union of prime ideals, then R_S denotes the ring of fractions of R with denominators in $R \setminus S$.

5.7 Lemma. Let I be an ideal of R and $I_S = f_S(I)R_S$ the extension of I to I_S. Then I_S consists of precisely those elements of R_S that admit a representation as a fraction with numerator in I:

$$I_S = \{\frac{i}{s} \in R_S \mid i \in I\}.$$

Proof. ($\subseteq$) is obvious. For ($\supseteq$), let $i \in I$, $s \in S$. Then

$$\frac{i}{s} = \frac{is^2}{s^3} = \frac{is}{s}\frac{s}{s^2} = f_S(i)\frac{s}{s^2} \in f_S(I)R_S.$$

$\square$

Note, however, that $\frac{r}{s} \in I_S$ does not imply $r \in I$, except in special cases such as:

5.8 Lemma. If P is a primary ideal of R with $P \cap S = \emptyset$ then $\frac{r}{s} \in P_S$ implies $r \in P$.

Proof. Let $\frac{r}{s} \in P_S$ with P primary. By 5.7, there exist $p \in P$ and $s' \in S$ with $\frac{r}{s} = \frac{p}{s'}$, and $ts'r = tsp$ for some $t \in S$. Since $ts' \in S$, no power of ts' is in P. Therefore $r \in P$. $\square$

Likewise, for every ideal J of R_S, $f_S^{-1}(J)$ is an ideal of R - the contraction of J to R. (If f_S is injective then $f_S^{-1}(J) = J \cap R$.) Also, if J is a prime [or primary] ideal then so is $f_S^{-1}(J)$.

5.9 Lemma. Let J be an ideal of R_S and I and ideal of R.

The inverse image of J under f_S consists precisely of all numerators that occur in fraction representations of elements of J:

$$f_S^{-1}(J) = \{r \in R \mid \exists s \in S\ \frac{r}{s} \in J\}.$$

Proof. ($\subseteq$) Let $r \in R$ with $f_S(r) \in J$. Then $\frac{s}{s^2} f_S(r) = \frac{s}{s^2}\frac{sr}{s} = \frac{r}{s} \in J$.

For ($\supseteq$), consider $r \in R$ with $\frac{r}{s} \in J$. Then $f_S(r) = \frac{sr}{s} = \frac{r}{s}\frac{s^2}{s} \in J$. $\square$

5.10 Theorem.

(a) For every ideal J of R_S, $(f_S^{-1}(J))_S = J$.

(b) Every ideal of R_S is of the form $I_S = \{\frac{i}{s} \in R_S \mid i \in I\}$ for some ideal I of R.

(c) $I_S = R_S$ iff $I \cap S \neq \emptyset$.

(d) A bijection between those prime ideals P of R with $S \cap P = \emptyset$ and all prime ideals of R_S is given by $P \mapsto P_S$ (for P a prime ideal of R); its inverse is $Q \mapsto f_S^{-1}(Q)$ (for Q a prime ideal of R_S).

(e) Like (d), with "prime" replaced by "primary".

5.11 Corollary. In terms of extension and contraction by $f_S\colon R \to R_S$, we have $J^{ce} = J$ for every ideal J of R_S and $P^{ec} = P$ for every primary ideal P of R with $P \cap S = \emptyset$.

5.12 Corollary. Let P be a prime ideal of R. Then every ideal of R_P is of the form $I_P = \{\frac{i}{s} \in R_P \mid i \in I\}$ for some ideal I of R with $I \subseteq P$. Via $Q \mapsto Q_P$ the prime ideals of R_P correspond bijectively to the prime ideals Q of R with $Q \subseteq P$. In particular, R_P is a local ring with maximal ideal P_P.

5.13 Definition. A ring containing exactly one maximal ideal is called a **local ring**. (Note, however, that many authors call such rings quasi-local, reserving the name local for Noetherian rings with a unique maximal ideal.)

5.14 Exercise. A ring is local if and only if the set of non-units forms an ideal.

5.15 Theorem (localization commutes with quotients). Let S be a multiplicative subset of R, I an ideal of R, and $\pi\colon R \to R/I$ the canonical projection then

$$(R/I)_{((S+I)/I)} \simeq R_S/I_S.$$

5.16 Easy exercise. The following is a well-defined ring-isomorphism:

$$\frac{(r+I)}{(s+I)} \mapsto \frac{r}{s} + I_S$$

5.17 Corollary. For every prime ideal P of R

$$(R/P)_{(0)} \simeq R_P/P_P.$$

5.18 Definition. If P is a prime ideal of R, we have seen that the quotient field of the integral domain R/P is the same as the residue field of the maximal ideal P_P of R_P. This field is called the **residue field** of P.

$***$ *Symbolic powers* $***$

5.19 Definition. Let P be a prime ideal of R. The n-th symbolic power of P is defined by $P^{(n)} = R \cap P^n R_P$.

The n-th symbolic power of P is the contraction back to R of the extension of P^n into R_P and thus contains P^n.

5.20 Proposition. Let P be a prime ideal of R and $n \in \mathbb{N}$.

(1) $P^{(n)} = (P_P)^n \cap R$

(2) $P^{(n)}$ is P-primary.

(3) If Q is a primary ideal then $P^n \subseteq Q \implies P^{(n)} \subseteq Q$.

(4) $P^{(n)} = P^n \iff P^n$ is primary.

(5) If M is a maximal ideal of R then $M^{(n)} = M^n$ for all $n \in \mathbb{N}$.

Proof. (2) Since P_P is a maximal ideal of R_P, $(P_P)^n$ is P_P-primary. By 10.18, $P^{(n)}$ is P-primary. (4) follows from (2) and (3); (5) follows from (4). $\square$