8. *Associated ideals*

*** *Zero-divisors and associated primes* ***

The following theorem holds for (not necessarily Noetherian) commutative rings:

8.1 Theorem (Prime Avoidance). Let R be a commutative ring and $P_1, \ldots, P_n$ ideals of R such that at most two of the P_i are not prime. Let I be a ring (possibly without a unit element) such that $I \subseteq \bigcup_{1 \le i \le n} P_i$. Then there exists an i with $S \subseteq P_i$.

Proof. Induction on n. Let $n=2$, $I \subseteq P_1 \cup P_2$, and suppose there exists $a_1 \in S \subseteq P_1$ and $a_2 \in S \subseteq P_2$. Then $a_1 + a_2$ is in I, but neither in P_1 nor in P_2, a contradition. Now let $n > 2$, $I \subseteq \bigcup_{1 \le i \le n} P_i$, but for every $1 \le k \le n$, I is not contained in the union of the P_i with $i \ne k$ (otherwise we are done by induction hypothesis). For every k pick a_k in I and not in the union of the P_i with $i \ne k$. Then $a_k \in P_k$. As $n > 2$, at least one of the P_i is prime, say P_1 is prime, and we set $b = a_1 + a_2 \ldots a_n$. Then b is in I, but not in any P_i, a contradiction $\square$

8.2 Definition. Let M be an R-module.

(i) For $m \in M$, the **annihilator** of m is $\mathrm{Ann}_R(m) := \{r \in R \mid rm = 0\}$.

(ii) $r \in R$ is a **zero-divisor** of M if $rm = 0$ for some non-zero $m \in M$.

(iii) The set of zero-divisors of M is denoted by $\mathcal{Z}(M)$, in other words,

$$\mathcal{Z}(M) = \bigcup_{\substack{m \in M \\ m \neq 0}} \mathrm{Ann}_R(m).$$

(iv) A prime ideal P of R is called an **associated prime ideal of** M if $P = \mathrm{Ann}_R(m)$ for some non-zero $m \in M$.

8.3 Lemma. Let M be a Noetherian R-module and N an R-submodule generated by the set S. Then N is generated by a finite subset of S.

Proof. Consider the (clearly non-empty) set of submodules of N generated by finite subsets of S. This set of submodules has a maximal element N', generated by $s_{i_1}, \ldots, s_{i_n} \in S$. For any $s \in S$, the submodule of N generated by s, $s_{i_1}, \ldots, s_{i_n}$ contains N' and is therefore equal to N' by maximailty of N'. We see that every $s \in S$ is already in N' and therefore $N = N'$. $\square$

8.4 Remark: For any commutative ring R and R-module M, the set $R \setminus \mathcal{Z}(M)$ of non-zerodivisors of M is a saturated multiplicative set. Therfore, by 10.3, its complement, $\mathcal{Z}(M)$, is a union of prime ideals. In the Noetherian case, we will see that only finitely many prime ideals are needed.

8.5 Proposition (Prime and maximal annihilators). Let R be a commutative ring and $M \neq 0$ an R-module.

(i) Every maximal annihilator of a non-zero element is prime.

(ii) If R is Noetherian then every annihilator of a non-zero element is contained in a maximal one.

(iii) If M is a Noetherian module then there are only finitely many maximal annihilators of non-zero elements of M.

(iv) If R is a Noetherian ring and M a Noetherian module, then $\mathcal{Z}(M)$ is a finite union of prime ideals, each of which is an annihilator of a non-zero element of M:

$$\mathcal{Z}(M) = \operatorname{Ann}_R(m_1) \cup \ldots \cup \operatorname{Ann}_R(m_n), \quad \operatorname{Ann}_R(m_i) \text{ prime, for } 1 \leq i \leq n$$

Proof.
(i) Let $I = \operatorname{Ann}_R(m)$ be maximal among annihilators of non-zero elements. $I \neq R$ since M is unitary and non-zero. Suppose $ab \in I$ and $b \notin I$. Then $bm \neq 0$ and $a \in \operatorname{Ann}_R(bm)$. Now $\operatorname{Ann}_R(m) \subseteq \operatorname{Ann}_R(bm)$ implies, by maximality of I, that $\operatorname{Ann}_R(m) = \operatorname{Ann}_R(bm)$ and therefore $a \in I$.
(ii) For $I = \operatorname{Ann}_R(m)$, consider the set $\mathcal{S}$ of annihilators of non-zero elements containing I (which is non-empty since $I \in \mathcal{S}$). Since R is Noetherian $\mathcal{S}$ has a maximal element which is clearly maximal among annihilators of non-zero elements and contains I.
(iii) Suppose $P_\lambda = \operatorname{Ann}_R(m_\lambda)$, $\lambda \in \Lambda$ are all the maximal annihilators of non-zero elements of M. Consider the submodule N of M generated by the m_λ, $\lambda \in \Lambda$. There exists a finite subset $m_{\lambda_1}, \ldots, m_{\lambda_n}$ of these generators that already generate N. For any m_μ with $\mu \in \Lambda$ there exist $r_1, \ldots, r_n \in R$, such that $m_\mu = r_1 m_{\lambda_1}, + \ldots + r_n m_{\lambda_n}$, which implies $P_{\lambda_1} \ldots P_{\lambda_n} \subseteq Ann_R(m_\mu)$. Since $Ann_R(m_\mu) = P_\mu$ is prime we have $P_{\lambda_i} \subseteq P_\mu$ for some $1 \leq i \leq n$. By maximality of P_{λ_i} it follows that $P_\mu = P_{\lambda_i}$. Therfore every $\operatorname{Ann}_R(m_\mu)$, $\mu \in \Lambda$ is one of $P_{\lambda_1}, \ldots, P_{\lambda_n}$.

(iv) follows from (i)–(iii). □

8.6 Theorem. Let R be a Noetherian ring and M a Noetherian R-module. If I

is a subring (possibly without a unit) of R contained in $\mathcal{Z}(M)$ then there exists a non-zero element $m \in M$ with $Im = \{0\}$.

Proof. By the previous lemma, $\mathcal{Z}(M)$ is a union of finitely many prime ideals of the form $\mathrm{Ann}_R(m)$. By prime avoidance 8.1, I is contained in a single $\mathrm{Ann}_R(m)$. $\square$

8.7 Corollary (Universal zero divisor). Let R be a Noetherian ring, and I an ideal (or non-unitary subring) consisting entirely of zero-divisiors of R. Then there exists $r \in R$ such that $ir = 0$ for all $i \in I$.

$* * *$ *Associated primes* $* * *$

8.8 Definition. If M is an R-module, an **associated prime** of M is a prime ideal P of R that is the annihilator of an element of M, $P = \mathrm{Ann}(m)$. The set of all associated primes of M is denoted by $\mathrm{Ass}_R(M)$, or $\mathrm{Ass}(M)$, when R is understood.

8.9 Definition. The associated primes of an ideal I of an integral domain R are the associated primes of the R-module R/I.

The apparent ambiguity of the definition of the associated primes of an ideal I is not a problem, as the only associated prime of the R-module I is (0) (in an intergal domain, $\mathrm{Ann}(r) = 0$ for every non-zero $r \in R$), so this is a concept we can disregard.

Let R be an integral domain with quotient field K and $k \in K$. Consider the "conductor ideal" $(R :_R k) = \{r \in R \mid rk \in R\}$. Note that $(R :_R k) = R$ iff $k \in R$. Also, if k is written as a fraction of elements of R, $k = \frac{a}{b}$ then $(R :_R k) = (Rb :_R a)$, which is the annihilator of (the residue class of) a in R/Rb. Therefore conductor ideals (of elements of the quotient field) that are prime are exactly the associated primes of non-zero principal ideals of R.