13. *A little point-set topology.*

$***$ *Open and closed sets* $***$

13.1 Definition. A subset τ of the power set $\mathcal{P}(X)$ of a set X is called a **topology** on X if the following axioms hold:

(O1) $\emptyset \in \tau$ and $X \in \tau$

(O2) $S, T \in \tau \implies S \cap T \in \tau$

(O3) $S_i \in \tau$ for all $i \in I$ (I an arbitrary index set) $\implies \bigcup_{i \in I} S_i \in \tau$

(X, τ) is then called a **topological space**. The members of τ are called **open sets**.

If (X, τ) is a topological space, then the complements of open sets, $A = X \setminus S$ with $S \in \tau$, are called **closed sets**. The collection of closed sets $\mathcal{C} = \{X \setminus O \mid O \in \tau\}$ satisfies

(C1) $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$.

(C2) $A, B \in \mathcal{C} \implies A \cup B \in \mathcal{C}$.

(C3) $A_i \in \mathcal{C}$ for all $i \in I$ (I an arbitrary index set) $\implies \bigcap_{i \in I} A_i \in \mathcal{C}$

Conversely, if $\mathcal{C} \subseteq \mathcal{P}(X)$ satisfys C1–C3, then the complements of elements of $\mathcal{C}$ form a topology on X, whose closed sets are precisely the elements of $\mathcal{C}$.

13.2 Example: Zariski topology on the spectrum of a ring. Let R be a commutative ring. A topology on the spectrum of R ($\operatorname{Spec}(R) = \{P \mid P$ a prime ideal of $R\}$) is defined by specifying its closed sets as sets of prime ideals of the form

$$V(I) = \{P \in \operatorname{Spec}(R) \mid P \supseteq I\},$$

for some ideal I of R.

There's nothing to prevent sets from being open and closed at the same time. Sets both open and closed are often called **clopen**.

13.3 Definition. Let (X, τ) be a topological space.

A collection $\mathcal{B}$ of open sets such that every open set is a union of elements of $\mathcal{B}$ is called a **basis** of the topology τ.

A collection $\mathcal{S}$ of open sets such that every open set is a union of finite intersections of elements of $\mathcal{S}$ is called a **subbasis** of the topology τ.

Topologies can be defined by specifying a basis or subbasis: If $\mathcal{B} \subseteq \mathcal{P}(X)$ is closed with respect to finite intersections, then the unions of (arbitrarily many) members of $\mathcal{B}$ form a topology on X, of which $\mathcal{B}$ is a basis.

If $\mathcal{S} \subseteq \mathcal{P}(X)$ is any collection of subsets of X, then arbitrary unions of finite intersections of members of $\mathcal{S}$ form a topology on X, of which $\mathcal{S}$ is a subbasis.

13.4 Example: Order Topology. If $(X, \leq)$ is a totally ordered set, then **order topology** on X is defined by specifying "open rays", i.e., sets of the form $(a, \infty) = \{x \in X \mid a < x\}$ and $(-\infty, b) = \{x \in X \mid x < b\}$, for $a, b \in X$ as a subbasis.

If $(X, \leq)$ has neither a maximal nor a minimal element, then "open intervals" $(a, b) = \{x \in X \mid a < x < b\}$ form a basis of order topology. (If X does have a maximal or minimal element, then open rays of the form (a, ∞), or $(-\infty, b)$, respectively, have to be added to the open intervals to get a basis.)

13.5 Definition. If (X, τ) is a topological space and Y a subset of X then Y inherits a topological structure from X (called **subspace topology**) through the convention: a subset U of Y is open (in Y) iff there exists an open subset O of X with $U = O \cap Y$.

If Y is an open subset of X then $U \subseteq Y$ is open in Y if and only it is open in X; if Y is a closed subset of X then $A \subseteq Y$ is closed in Y if and only it is closed in X.

13.6 Remark. If $(X, \leq)$ is a totally ordered set and $Y \subseteq X$, then Y inherits a topology from the order topology of X, and at the same time Y inherits an order relation from X which makes $(Y, \leq)$ a totally ordered set, for which order topology may be defined. These two topologies on Y in general do not agree. (Examples can be found among subsets of the real numbers.)

$* * *$ *Neighborhoods and neighborhood bases* $* * *$

Perhaps a more intuitive approach to topology is through neighborhoods of a point, which (as sets containing an open ball around the point) are already familiar from the study of metric spaces.

13.7 Definition. $(*)$ Let (X, τ) be a topological space and $p \in X$. A neighborhood of p is a set U such that there exists an open set O with $p \in O \subseteq U$. The set of all neighborhoods of a point p is called the **neighborhood filter** of p. We will denote in by $\mathcal{U}(p)$.

13.8 Remark. Let (X,τ) be a topological space and $p \in X$. The neighborhood filter of p has the properties

(U1) $\forall U \in \mathcal{U}(p)\ \ p \in U$.

(U2) $U,\, V \in \mathcal{U}(p) \implies U \cap V \in \mathcal{U}(p)$

(U3) $U \in \mathcal{U}(p)$ and $V \supseteq U \implies V \in \mathcal{U}(p)$

(U4) $\forall U \in \mathcal{U}(p)\, \exists V \in \mathcal{U}(p)\, \forall v \in V\ \ U \in \mathcal{U}(v)$.

Also,

$$(*) \qquad O \in \tau \iff \forall p \in O\ \ \exists U \in \mathcal{U}(p)\ \ U \subseteq O$$

Conversely, given a set X and for each $p \in X$ a set $\mathcal{U}(p) \subseteq \mathcal{P}(X)$ such that U1–U4 holds, we can define a topolgy τ on X by $(*)$, and, what is more, the neighborhood filter of each point in the resulting topology τ is exactly the $\mathcal{U}(p)$ we started out with.

As with metric spaces, it suffices to know a system of "basic" neighborhoods of a point – with the property that every neighborhood contains one of them – to know all neighborhoods.

13.9 Definition. Let (X,τ) be a topological space. A collection $\mathcal{B}(p) \subseteq \mathcal{U}(p)$ of neighborhoods of p is called a **neighborhood basis** of p if $\forall U \in \mathcal{U}(p) \exists B \in \mathcal{B}(p)\ B \subseteq U$.

If (X,τ) is a topological space, and for each $p \in X$, $\mathcal{B}(p)$ is a neighborhood basis, then

(B1) $\forall B \in \mathcal{B}(p)\ \ p \in B$.

(B2) $U,\, V \in \mathcal{B}(p) \implies \exists B \in \mathcal{B}(p)\ \ B \subseteq U \cap V$

(B3) $\forall U \in \mathcal{B}(p)\, \exists V \in \mathcal{B}(p)\, \forall v \in V\, \exists B \in \mathcal{B}(v)\ B \subseteq U$).

Also,

$$(**) \qquad O \in \tau \iff \forall p \in O\ \ \exists B \in \mathcal{B}(p)\ \ B \subseteq O$$

Conversely, if we are given for every $p \in X$ a collection $\mathcal{B}(p)$ of subsets of X satisfying B1–B3, we can define a topology on X by $(**)$, and in this topology $\mathcal{B}(p)$ will be a neighborhood basis of p.

13.10 Definition. A topological space in which every point has a countable neighborhood basis is said to satisfy the **first countability axiom**.

13.11 Example: Let (X, d) be a metric space, and $B_\varepsilon(p) = \{x \in X \mid d(p, x) < \varepsilon\}$ (for $\varepsilon > 0$, $p \in X$) the open ball of radius ε around p. If we define $\mathcal{B}(p)$ as the set of all $B_\varepsilon(p)$ with $\varepsilon > 0$, then B1–B3 hold. In other words, every metric induces a topology in which the open ε-balls with center p form a neighborhood basis of p. Actually, countably many balls $B_{\frac{1}{n}}(p)$, $n \in \mathbb{N}$, already form a neighborhood basis of p in this topology. We see that every metric space satisfies the first countability axiom.

13.12 Example: I-adic topology Let R be a commutative ring and I an ideal of R. I-adic topology on R is defined by specifying a neighborhood base of $r \in R$:

$$\mathcal{B}(r) = \{r + I^n \mid n \in \mathbb{N}\}.$$

Note that these base neighborhoods are both open and closed.

$* * *$ *Closure and interior* $* * *$

13.13 Definition. Let A be a subset of a topological space X. The **closure** of A, denoted $\bar{A}$, is defined to be the intersection of all closed sets containing A.

Clearly, $\bar{A}$ is a closed set containing A and every closed set B that contains A also contains $\bar{A}$.

The **closure operator** $A \mapsto \bar{A}$ on $\mathcal{P}(X)$ has the properties:

(A1) $\bar{\emptyset} = \emptyset$.

(A2) $A \subseteq \bar{A}$.

(A3) $\bar{\bar{A}} = \bar{A}$.

(A4) $\bar{A} \cup \bar{B} = \overline{A \cup B}$.

Conversely, any operator $A \mapsto \bar{A}$ on the power set of a set X with properties A1–A4 may be used to define a topology on X by declaring the closed sets to be precisely the sets of the form $\bar{A}$ for some $A \in \mathcal{P}(X)$. (To see that these sets satisfy C3, first note that A4 implies $A \subseteq C \implies \bar{A} \subseteq \bar{C}$. From this we get that $\overline{\bigcap_{i \in I} \bar{A}_i} \subseteq \bar{A}_i$ for all $i \in I$, i.e., $\overline{\bigcap_{i \in I} \bar{A}_i} \subseteq \bigcap_{i \in I} \bar{A}_i$ and the reverse inclusion is just A2.)

13.14 Definition. Let (X, τ) be a topological space and $A \in \mathcal{P}(X)$ then the **interior** of A, denoted A°, is the union of all open sets contained in A.

Clearly, A° is open and $A^\circ \subseteq A$. Also, $(X \setminus A)^\circ = X \setminus \bar{A}$. (Note that the interior of a non-empty set may be empty.)

*** *Boundry points, accumulation points and such* ***

The following properties of the closure of a set (which we defined as the intersection of all closed sets containing it) and the interior of a set (which we defined as the union of all open sets contained in it) are often used as definitions:

13.15 Lemma.

- $\bar{A}$ consists of precisely those points $p \in X$ such that $U \cap A \neq \emptyset$ for every neighborhood $U \in \mathcal{U}(p)$ and
- A° consists of precisely those points $p \in X$ such that there exists a neighborhood $U \in \mathcal{U}(p)$ with $U \subseteq A$.

13.16 Definition. Let (X, τ) be a topological space, $p \in X$ and $A \subseteq X$ then

- p is called a **boundary point** of A if for every neighborhood $U \in \mathcal{U}(p)$ both $A \cap U \neq \emptyset$ and $(X \setminus A) \cap U \neq \emptyset$. The set of all boundary points of A is the **boundary** of A, denoted by δA.
- p is called an **interior point** of A if $p \in A^\circ$, i.e., if there exists a neighborhood $U \in \mathcal{U}(p)$ with $U \subseteq A$.
- p is called an **accumulation point** of A if every neighborhood of p contains an element of A other than p.
- p is called an **isolated point** of A if there exists a neighborhood U of p with $U \cap A = \{p\}$.

Note that interior points of A and isolated points of A are necessarily in A, while boundary points and accumulation points may or may not belong to A.

By purely logical arguments we see that every set A induces a partition of X into three disjoint parts in two different ways ($\dot\cup$ denotes disjoint union):

(i) $X = A^\circ \,\dot\cup\, \delta A \,\dot\cup\, (X \setminus A)^\circ$ and

(ii) $X = \{\text{isolated points of } A\} \dot\cup \{\text{accumulation points of } A\} \dot\cup (X \setminus A)^\circ$.

Also,

(iii) $\bar{A} = A^\circ \,\dot\cup\, \delta A$ and

(iv) $\bar{A} = \{\text{isolated points of } A\} \,\dot\cup\, \{\text{accumulation points of } A\}$

The last two partitions of $\bar{A}$ follow from the characterization of $\bar{A}$ as the set of those p such that $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}(p)$. They are incomparable in the sense that all four combinations of belonging to one set of one partition and one set of the other are possible, an accumulation point of A can be either an interior point or a boundary point of A, etc. Also, we have seen that

(v) $\bar{A} = A \,\cup\, \delta A$ and

(vi) $\bar{A} = A \cup \{\text{accumulation points of } A\}$

Unlike (iii) and (iv), the unions (v) and (vi) are in general not disjoint.

$*\,*\,*$ *Continuous functions* $*\,*\,*$

13.17 Definition. Let (X, τ) and (Y, τ') be topological spaces. A function $f\colon X \to Y$ is called **continuous** if $f^{-1}(O)$ is open for every open set $O \subseteq Y$.

In terms of neighborhoods, a function $f\colon X \to Y$ is continuous, if and only if for every $x \in X$, for every $U \in \mathcal{U}(f(x))$ there exists a $V \in \mathcal{U}(x)$ with $f(V) \subseteq U$. The familiar ε-δ-definition of continuous functions is easily seen to be the specialization to metric spaces of this topological characterization.

13.18 Definition. Let (X, τ) and (Y, τ') be topological spaces. A function $f\colon X \to Y$ is called **open** if $f(O)$ is open for every open set $O \subseteq X$.

A bijective function that is both open and continuous is called a **homeomorphism**.

A topology τ_1 on X is called **stronger** (or **finer**) than another topology τ_2 on the same set X if $\tau_1 \supseteq \tau_2$ (every τ_2-open set is τ_1-open); τ_2 is then called **weaker** or **coarser** than τ_1. Two trivial topologies exist on every set X: **discrete topology** $\tau = \mathcal{P}(X)$ (the finest topology on X) and **chaotic topology** $\tau = \{\emptyset, X\}$ (the coarsest topology on X).

If τ_1 and τ_2 are two topologies on a set X then τ_1 is stronger than τ_2 iff $\mathrm{id}_X : (X, \tau_1) \to (X, \tau_2)$ is continuous; τ_1 is weaker than τ_2 iff $\mathrm{id}_X : (X, \tau_1) \to (X, \tau_2)$ is open.