2. *Modules*

2.1 Definition. Let R be a commutative ring. An Abelian group $(A, +)$ together with a "scalar" multiplication $\cdot : R \times A \to A$ by elements of R is called an **R-module**, if the following conditions hold for all $r,\ s \in R$ and $a,\ b \in A$:

$(r+s)a = ra + sa$

$(rs)a = r(sa)$

$r(a+b) = ra + rb$

In addition, we will always assume that modules are **unitary**, which means that multiplication by $1 \in R$ is the identity map on A, i.e., for all $a \in A$, $1_R a = a$.

2.2 Definition. Given an R-module A, the **annihilator** of a subset B of A is $\mathrm{Ann}_R(B) = \{r \in R \mid \forall b \in B\ \ rb = 0\}$.

Clearly, $\mathrm{Ann}_R(B)$ is an ideal of R for any B. Of particular importance is the annihilator of the whole module. If $\mathrm{Ann}_R(A) = I$ for an R-module A then A can be regarded as an R/I-module by defining $(r+I)a := ra$.

2.3 Definition. An R-module A is called **faithful** if $\mathrm{Ann}_R(A) = (0)$.

2.4 Definition. A (possibly non-commutative) ring A is called an **R-algebra**, if $(A, +)$ is an R-module and the multiplication of the ring A interacts peacefully with the scalar multiplication as follows: for all $r \in R$ and $a,\ b \in A$

$$r(ab) = (ra)b = a(rb).$$

Examples of R-algebras include the polynomial ring $R[x]$ and the ring $M_n(R)$ of $n \times n$ matrices with entries in R. More generally, every ring T of which R is a subring is an R-algebra (by restriction of the ring-multiplication on T to $\cdot : R \times T \to T$).

Because of $rb = (r1_A)b$, an R-algebra A is faithful if and only if $\mathrm{Ann}(1_A) = (0)$, and therefore every faithful R-algebra A admits an embedding of R by $r \mapsto r1_A$. If we call $\tilde{R}$ the copy of R thus embedded in A, then, again by $rb = (r1_A)b$, the scalar multiplication of the R-algebra A is just the restriction of the ring multiplication of A to $\tilde{R} \times A$.