5. *Rings of Quotients and Localization*

A **multiplicatively closed** subset of a ring is a set $S \neq \emptyset$ such that $st \in S$ whenever $s, t \in S$.

5.1 Definition. If S is a multiplicatively closed subset of a commutative ring R, then the **ring of fractions with denominators in** S, $(S^{-1}R, +, \cdot)$ and the canonical homomorphism $f_S \colon R \to S^{-1}R$ are defined as follows:

$S^{-1}R$ is the set of equivalence classes on $S \times R$ under the equivalence relation

$$(s, r) \sim (s', r') \;:\Longleftrightarrow\; \exists t \in S \;\; tsr' = ts'r.$$

The equivalence class of (s, r) is denoted by $\frac{r}{s}$. Addition and multiplication are defined by

$$\frac{r}{s} + \frac{r'}{s'} = \frac{s'r + sr'}{ss'} \qquad \text{and} \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

and the canonical homomorphism $f_S \colon R \to S^{-1}R$ by $f_S(r) = \frac{rs}{s}$ (for any $s \in S$).

A lot of facts have to be checked before the above definition can be called valid, for instance that $\sim$ is really an equivalence relation, that addition and multiplication are well-defined (i.e. the resulting equivalence class depends only in the equivalence classes of the arguments and not on the numerator and denominator of the fractions representing them), that $S^{-1}R$ satisfies the ring axioms, and that f_S is well-defined and a ring homomorphism. These verifications are easy, if tedious. In doing them, one also stumbles upon the following properties of $S^{-1}R$:

(1) $S^{-1}R$ is a commutative ring with zero $\frac{0}{s}$ and unit element $\frac{s}{s}$ (for any $s \in S$).
(2) $\frac{r}{s} = \frac{r'}{s'}$ iff there exists a $t \in S$ with $trs' = tr's$.
(3) If S contains no zero-divisors, then $\frac{r}{s} = \frac{r'}{s'}$ iff $rs' = r's$.
(4) For all $r \in R$ and $s, t \in S$ $\quad \frac{r}{s} = \frac{rt}{st}$.
(5) $\frac{r}{s} = 0$ iff there exists a $t \in S$ with $rt = 0$.
(6) $\frac{r}{s}$ is a unit iff there exists a $t \in S$ with $rt \in S$.
(7) For every $s \in S$ $\quad f_S(s)$ is a unit in $S^{-1}R$.
(8) $\mathrm{Ker} f_S = \{r \in R \mid \exists t \in S \; rt = 0\}$.
(9) f_S is injective iff S contains no zero-divisor.
(10) If $0 \in S$ then $S^{-1}R = \{0\}$.

Recall that a multiplicatively closed set S is **saturated**, if S contains all divisors of every $s \in S$. The saturation of a multiplicatively closed set S is $\bar{S} = \{t \in R \mid \exists r \in R \;\; rt \in S\}$.

5.2 Lemma. Let S be a multiplicatively closed subset of R and $\bar{S}$ its saturation. Then $\varphi: S^{-1}R \to \bar{S}^{-1}R$ given by $\varphi(\frac{r}{s}) = \frac{r}{s}$ is an isomorphism of rings.

Proof. φ is well-defined and a ringhomomorphism; to see injectivity, suppose $\frac{r}{s} \in \mathrm{Ker}\varphi$, then $rt = 0$ for some $t \in \bar{S}$. t divides some $t' \in S$, such that $t'r = 0$ and therefore $\frac{r}{s} = 0$ in $S^{-1}R$. For surjectivity, consider $\frac{r}{t}$ with $r \in R$ and $t \in \bar{S}$. For some $s \in S$ and $w \in R$ we have $s = tw$ (and thus $w \in \bar{S}$) and $\frac{r}{t} = \frac{rw}{tw} = \frac{rw}{s} = \varphi(\frac{rw}{s})$. □

By the previous lemma and Lemma 10.2, all rings of quotients of R are of the form form $S^{-1}R$ with $S = R \setminus \bigcup_{P \in \mathcal{P}} P$ for a set $\mathcal{P}$ of prime ideals of R.

5.3 Definition. Important examples of rings of fractions:

- If S is the set of all non-zero-divisors of R, then $S^{-1}R$ is called the **complete ring of fractions of** R; a special case of this is the **quotient field** of an integral domain, which is $S^{-1}R$ with $S = R \setminus \{0\}$.
- If $S = R \setminus P$, P a prime ideal of R, then $S^{-1}R$ is denoted R_P and called the **localization of** R **at** P. The ideal $S^{-1}I$ of R_P is denoted I_P or IR_P.

The importance of the construction lies in the relationship between ideals of R and ideals of $S^{-1}R$. We already know that for every ideal I of R, $f_S(I)S^{-1}R$ is an ideal of $S^{-1}R$ - the extension of I into $S^{-1}R$. It is denoted by $S^{-1}I$ and consists of all elements of $S^{-1}R$ that admit a representation as a fraction with numerator in I:

5.4 Lemma.

$$S^{-1}I = \{\frac{i}{s} \in S^{-1}R \mid i \in I\}.$$

Note, however, that $\frac{r}{s} \in S^{-1}I$ does not imply $r \in I$, exept in special cases such as:

5.5 Lemma. If P is a primary ideal of R with $P \cap S = \emptyset$ then $\frac{r}{s} \in S^{-1}P$ implies $r \in P$.

Likewise, for every ideal J of $S^{-1}R$, $f_S^{-1}(J)$ is an ideal of R - the contraction of J into R. (If f_S is injective then $f_S^{-1}(J) = J \cap R$.) Also, if J is a prime [or primary] ideal then so is $f_S^{-1}(J)$. It is easy to see that $f_S^{-1}(J)$ consists precisely of all numerators that occur in fraction representations of elements of J:

5.6 Lemma. Let J be an ideal of $S^{-1}R$. Then

$$f_S^{-1}(J) = \{r \in R \mid \exists s \in S\ \frac{r}{s} \in J\}.$$

In particular, for every ideal I of R, $f_S^{-1}(S^{-1}I)$ is the set of numerators that occur in fractions representing elements of $S^{-1}I$.

5.7 Theorem.

(a) $S^{-1}I = S^{-1}R$ iff $I \cap S \neq \emptyset$.

(b) For every ideal J of $S^{-1}R$, $S^{-1}f_S^{-1}J = J$; in particular, every ideal of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R.

(c) A bijection between the prime ideals P of R with $S \cap P = \emptyset$ and all prime ideals of $S^{-1}R$ is given by $P \mapsto S^{-1}P$ (for P a prime ideal of R), its inverse is $Q \mapsto f_S^{-1}Q$ (for Q a prime ideal of $S^{-1}R$).

(d) Like (c), with "prime" replaced by "primary".

Rephrasing this in terms of extension and contraction, we have $J^{ce} = J$ for every ideal J of $S^{-1}R$ and $P^{ec} = P$ for every primary ideal P of R with $P \cap S = \emptyset$.

5.8 Corollary. Let P be a prime ideal of R. Then every ideal of R_P is of the form I_P for some ideal I of R with $I \subseteq P$ and via $Q \mapsto Q_P$ the prime ideals of R_P correspond bijectively to the prime ideals Q of R with $Q \subseteq P$. In particular, P_P is a maximal ideal of R_P, and its only maximal ideal.

By the above theorem, the localization of R at a prime ideal P contains a unique maximal ideal, P_P.

5.9 Definition. A ring containing exactly one maximal ideal is called a **local ring**. (Note, however, that many authors call such rings quasi-local, reserving the name local for Noetherian rings with a unique maximal ideal.)

5.10 Lemma. A ring is local if and only if the set of non-units forms an ideal.

Proof. Easy. □

5.11 Theorem(localization commutes with quotients). Let S be a multiplicative subset of R, I an ideal of R, and $\pi: R \to R/I$ the canonical projection then

$$(\pi(S))^{-1}(R/I) \simeq S^{-1}R/S^{-1}I.$$

5.12 Corollary. For every prime ideal P of R

$$((R/P) \setminus \{0 + P\})^{-1}R/P \simeq R_P/P_P.$$

In other words, if P is a prime ideal of R then the quotient field of the integral domain R/P is the same as the residue field of the maximal ideal P_P of R_P. This field is called the **residue field** of P.

$* * *$ *Symbolic powers* $* * *$

5.13 Definition. Let P be a prime ideal of R. The n-th symbolic power of P is defined by $P^{(n)} = R \cap P^n R_P$.

The n-th symbolic power of P is the contraction back to R of the extension of P^n into R_P and thus contains P^n.

5.14 Proposition. Let P be a prime ideal of R and $n \in \mathbb{N}$.

(1) $P^{(n)}$ is P-primary.

(2) If Q is a primary ideal then $P^n \subseteq Q \implies P^{(n)} \subseteq Q$.

(3) $P^{(n)} = P^n \iff P^n$ is primary.

5.15 Corollary. If M is a maximal ideal of R then $M^{(n)} = M^n$ for all $n \in \mathbb{N}$.