1. *Extension-Contraction and the Spectrum of a Polynomial Ring*

∗∗∗ *Extension and Contraction of Ideals* ∗∗∗

If $R \subseteq S$ are commutative rings then every ideal I of R defines an ideal of S by **expansion**: $I^e = IS$, and every ideal J of S defines an ideal of R by contraction $J^e = J \cap R$. More generally, if $f\colon R \to S$ is a ring-homomorphism, then every $I \trianglelefteq R$ defines an ideal $I^c = f(I)S$ of S and every $J \trianglelefteq S$ an ideal $J^e = f^{-1}(J)$ of R. (When f is understood, I^e is often written as IS and J^e as $J \cap R$ by abuse of notation even if f is not injective.)

Contraction preserves the property of being a prime ideal (or a primary ideal); extension, in general, does not.

Clearly, for ideals A, B of R and C, D of S, $A \subseteq B \Rightarrow A^c \subseteq B$ and $C \subseteq D \Rightarrow C^e \subseteq D^e$; and $A^{ce} \subseteq A$ and $D^{ec} \supseteq D$.

Therefore, for every ring homomorphism $f\colon R \to S$, extension and contraction give a bijective correspondence between ideals of R of the form $J^c = J \cap R$ (J an ideal of S) and ideals of S of the form $I^e = IS$ (I an ideal of R). In two noteable cases, all ideals of S are of the form I^e:

- Let H be an ideal of R and $\pi : R \to R/H$, $\pi(r) = r + H$, the canonical projection. Then extension and contraction induce a bijection between all ideals of R/H and the ideals of R containing H. In both directions of this bijection the properties of being a prime ideal, a primary ideal, or a G-ideal are preserved. Note that for a subset S of R/H $\pi^{-1}(S) = \bigcup_{r+H \in S} r + H$.
- Let S be a multiplicative subset of R, $S^{-1}R$ the ring of quotients with denomiators in S and $f\colon R \to S^{-1}R$ the canonical map $f(r) = r/1$ (if R has a unit element, or $f(r) = rs/s$ otherwise). Recall that f is injective whenever S contains no zero-divisors. Then extension and contraction induce a bijection between all ideals of $S^{-1}R$ and those ideals of R that are disjoint from S. Most importantly, if $S = R \subseteq P$, P a prime ideal, then the bijection is between all ideals of R_P and the ideals of R contained in P.

Recall that for I an ideal of R, the rings $R[x]/I[x]$ and $(R/I)[x]$ are canonically isomorphic via $a_0 + a_1x + \ldots + a_nx^n + I[x] \mapsto (a_0 + I) + (a_1 + I)x + \ldots + (a_n + I)x^n$.

∗∗∗ *Spectrum of a Polynomial Ring* ∗∗∗

Let R be a polynomial ring and P a prime ideal of R. Then every prime ideal Q of $R[x]$ with $Q \cap R \supseteq P$ contains $P[x]$ and is therefore of the form $\pi^{-1}(\bar{Q})$, where $\pi\colon R[x] \to R[x]/P[x] = (R/P)[x]$ is the canonical projection and $\bar{Q}$ is a prime ideal of $(R/P)[x]$. Those Q among them with $Q \cap R = P$ correspond to those $\bar{Q}$ for which $\bar{Q} \cap (R/P) = (0 + P)$.

Let D be a domain with quotient field K. Then every prime ideal Q of $D[x]$ with $Q \cap D = (0)$ is of the form $\bar{Q} \cap D[x]$, where $\bar{Q}$ is a prime ideal of $K[x]$ (and thus $\bar{Q} = (f)$, where f is an irreducible polynomial of $K[x]$.

To summarize: to determine for a given prime ideal P of R all prime ideals Q of $R[x]$ with $Q \cap R = P$: consider the polynomial ring $K_P[x]$, where K_P is the quotient field of R/P, i.e. $K_P = (R/P \setminus \{0+P\})^{-1} R/P$ for each monic irreducible $f \in K_P[x]$, consider all polynomials of $R/P[x]$ that are divisible by f in $K_P[x]$, i.e. all products fg with $g \in K_P[x]$, such that $fg \in R/P[x]$. Then the collection of all polynomials of $D[x]$ whose residue class (under the projection of coefficients from R to R/P is of the form fg (for some $g \in K_P[x]$) is a prime ideal of $D[x]$ and all prime ideals Q of $D[x]$ with $Q \cap R = P$ are of this form.